INTRODUCTION TO INTEGRAL GEOMETRY IN RIEMANNIAN HOMOGENEOUS SPACES

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ABSTRACT. In these talks, we shall study fundamentals of integral geometry in Riemannian homogeneous spaces, especially kinematic formulae and their applications.

1. INTRODUCTION

In the last century, the branch of integral geometry was initiated from the idea of geometric probability by Blaschke, Chern, Santaló and other geometers. (Santaló's book [16] is a good reference for the history and introduction to integral geometry.) Let X be a set of certain geometric objects, such as points, lines, motions and so on. When we consider a geometric quantity for each element of X, it can be regarded as a function on X. We shall give a measure on X, and evaluate the integral of that function on X as the "average" of that geometric quantity. This integration sometimes leads to geometric consequences. One of the most important problem to do this is how to define an appropriate measure on X. When X is a homogeneous space, a method of Lie groups provides invariant measures on X. In this series of talks we shall study such geometric integral formulae in Riemannian homogeneous spaces.

Let G/K be a homogeneous space with an invariant Riemannian metric. Consider two compact submanifolds M and N of G/K, one fixed and the other moving under the action of $g \in G$. Then it becomes a measurable function on G if we assign an integral invariant $I(M \cap gN)$ of submanifold $M \cap gN$ for each $g \in G$. It is called the kinematic formula that represents the equality between the integral

$$\int_G I(M \cap gN) d\mu_G(g) \tag{1.1}$$

and some geometric invariants of M and N, where $d\mu_G$ is the invariant measure of G. In particular, it is called Poincaré formula if we take $\mathrm{vol}(M \cap gN)$ as an integral invariant in (1.1). Furthermore, Chern [3] and Federer [4] obtained a kinematic formula for integral invariants which appear in coefficients of the Weyl tube formula and the generalized Gauss-Bonnet formula. In 1993 Howard [5] defined integral invariants induced from an invariant homogeneous polynomial of the second fundamental form of $M \cap gN$, and he achieved more general kinematic formula. He put the Poincaré formula, the Chern-Federer formula etc. into a uniform shape.

The kinematic formulae are interesting in their own forms, moreover they have been applied to several geometric problems. For example, Zhou [20] was obtained an extension of Chen's formula, and applied it to describe a sufficient condition for Hadwiger's containment problem in higher dimensions. The Poincaré formula has been applied to show some volume minimizing properties of certain submanifolds (see [6], [8], [9] and [18]).

2. Integral geometry in the plane

We begin with classical geometric integral formulae in the plane. One of the most fundamental ones is the kinematic formula of intersection numbers of two curves in the plane.

Let us consider the set $I(\mathbb{R}^2)$ of all isometries of $\mathbb{R}^2$ which preserve the orientation. An element of $I(\mathbb{R}^2)$ can be expressed as

$$T(\phi, u, v) : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}.$$

A triad (ϕ, u, v) of real numbers provides a local coordinate and the differentiable manifold structure on $I(\mathbb{R}^2)$. As a Lie group, $I(\mathbb{R}^2)$ is isomorphic to the semi-direct product $SO(2) \ltimes \mathbb{R}^2$. We give a Riemannian metric on $I(\mathbb{R}^2)$ by the product metric of $SO(2)$ and $\mathbb{R}^2$. Then it is known the following theorem.

Theorem 2.1 (Poincaré formula)**.** *For two curves c_0 and c_1 in $\mathbb{R}^2$*

$$\int_{I(\mathbb{R}^2)} \#(c_0 \cap gc_1) d\mu(g) = 4\,\mathrm{Length}(c_0)\mathrm{Length}(c_1)$$

holds. Here $d\mu$ denotes the Riemannian measure on $I(\mathbb{R}^2)$.

From the Poincaré formula, we can obtain some other kinematic formulae. Let $S(\boldsymbol{x}, r)$ denote a circle of radius r centered at $\boldsymbol{x}$ in $\mathbb{R}^2$. For a fixed r, the set of all circles of radius r in $\mathbb{R}^2$ is identified with $\mathbb{R}^2$.

Theorem 2.2. *For a curve in $\mathbb{R}^2$*

$$\int_{\mathbb{R}^2} \#(c \cap S(\boldsymbol{x}, r)) d\mu(\boldsymbol{x}) = 4r\,\mathrm{Length}(c)$$

holds.

Proof. Apply the Poincaré formula for a circle $c_1 = S(O, r)$ centered at the origin O. Then we have

$$\begin{aligned} 4 \cdot 2\pi r \cdot \mathrm{Length}(c) &= \int_{I(\mathbb{R}^2)} \#(c \cap gS(O, r)) d\mu(g) \\ &= \int_{\mathbb{R}^2} \int_0^{2\pi} \#(c \cap T(\phi, u, v) \cdot S(O, r)) d\phi\, du dv \\ &= 2\pi \int_{\mathbb{R}^2} \#(c \cap S((u, v), r)) du dv, \end{aligned}$$

because $S(O, r)$ is invariant under the rotation. □

Next we shall consider the set $L(\mathbb{R}^2)$ of all lines in $\mathbb{R}^2$. An element of $L(\mathbb{R}^2)$ can be expressed by a pair (r, θ) of real numbers as

$$l(r, \theta) = \{(x, y) \in \mathbb{R}^2 \mid x\cos\theta + y\sin\theta = r\}.$$

Then, (r, θ) provides a local coordinate and the differentiable manifold structure on $L(\mathbb{R}^2)$. Actually $L(\mathbb{R}^2)$ is diffeomorphic to Möbius strip, therefore is non-orientable. We give a Riemannian metric g on $L(\mathbb{R}^2)$ which is locally isometric to $\mathbb{R}^2$, that is, $g = dr^2 + d\theta^2$. Then similarly with Theorem 2.2 we can obtain the following theorem from the Poincaré formula, however a line has infinite length.

Theorem 2.3 (Crofton formula)**.** *For a curve* c *in* $\mathbb{R}^2$

$$\int_{L(\mathbb{R}^2)} \#(c \cap l) d\mu(l) = 2\,\mathrm{Length}(c)$$

holds. Here $d\mu$ *denote the Riemannian measure on* $L(\mathbb{R}^2)$*.*

For a closed curve c in $\mathbb{R}^2$, let $d_c(\theta)$ denote the width of c with respect to the direction $(\cos\theta, \sin\theta)$. Then from the Crofton formula, we get the following.

Theorem 2.4 (Barbier's theorem)**.** *For a closed convex curve* c *in* $\mathbb{R}^2$

$$\int_0^\pi d_c(\theta) d\theta = \mathrm{Length}(c)$$

holds. Hence, if c *has constant width* d*, then* $\mathrm{Length}(c) = \pi d$*.*

Proof. From the Crofton formula,

$$2\,\mathrm{Length}(c) = \int_{L(\mathbb{R}^2)} \#(c \cap l) d\mu(l) = \int_0^\pi \int_{-\infty}^\infty \#(c \cap l(r, \theta)) dr\, d\theta.$$

For a fixed θ, there exists a closed interval $[r_\theta, r_\theta + d_c(\theta)]$ so that $l(r, \theta)$ meets c if and only if $r \in [r_\theta, r_\theta + d_c(\theta)]$. Since c in convex, $l(r, \theta)$ has exactly two intersection points with c for $r \in (r_\theta, r_\theta + d_c(\theta))$. Therefore we have

$$2\,\mathrm{Length}(c) = 2 \int_0^\pi \int_{r_\theta}^{r_\theta + d_c(\theta)} dr\, d\theta = 2 \int_0^\pi d_c(\theta) d\theta.$$

□

We also know the following remarkable kinematic formula.

Theorem 2.5 (Blaschke formula)**.** *Let* c_0 *and* c_1 *be simply closed curves in* $\mathbb{R}^2$*, and let* D_0 *and* D_1 *be domains encircled by* c_0 *and* c_1 *respectively. We denote by* $\chi(D)$ *the number of connected components of a domain* D *in* $\mathbb{R}^2$*. Then*

$$\int_{I(\mathbb{R}^2)} \chi(D_0 \cap gD_1) d\mu(g) = 2\pi(\mathrm{Area}(D_0) + \mathrm{Area}(D_1)) + \mathrm{Length}(c_0)\mathrm{Length}(c_1)$$

holds.

3. Kinematic formulae in Riemannian homogeneous spaces

In this section we shall study the general theory of the kinematic formulae in Riemannian homogeneous spaces due to Howard. Refer to his paper [5] for details.

Let G be a Lie group and K a compact subgroup of G. We assume that G has a left invariant metric that is also right invariant under K. Then a homogeneous space G/K has an invariant metric. Throughout this paper, we also assume that G is unimodular, that is $|\det \mathrm{Ad}(g)| = 1$ for any $g \in G$. We denote by $T = T_o(G/K)$ the tangent space of G/K at the origin o. Let V be a linear subspace of T. A submanifold M of G/K is said to be of type V if and only if for each $x \in M$ there exists a $g_x \in G$ such that $(g_x)_* V = T_x M$.

3.1. **Generalized Poincaré formula.** Let E be an n-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. For two vector subspaces V and W of dimensions p and q in E with $p + q \leq n$, we take orthonormal bases $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ of V and W respectively. Then we define $\sigma(V, W)$, the angle between V and W, by

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|,$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = \det(\langle x_i, x_j \rangle).$$

This definition is independent of the choice of orthonormal bases.

Now let us define the angle between subspaces of tangent spaces of a Riemannian homogeneous space G/K. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$ we define $\sigma_K(V, W)$, the angle between V and W, by

$$\sigma_K(V, W) = \int_K \sigma((g_x)_*^{-1} V,\ k_*^{-1} (g_y)_*^{-1} W) d\mu_K(k),$$

where g_x and g_y are elements of G so that $g_x o = x$ and $g_y o = y$. We note that this definition is independent of the choice of g_x and g_y.

With this notation, the generalized Poincaré formula for homogeneous spaces can be stated as follows:

Theorem 3.1 (Howard [5]). *Let M and N be submanifolds of G/K with* $\dim M + \dim N \geq \dim(G/K)$. *Then*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \int_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M \times N}(x, y) \tag{3.1}$$

holds.

Equality (3.1) implies that if M is a submanifold of G/K of type V and N of type W for some subspaces V and W of T, then σ_K is a constant function on $M \times N$ and

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \sigma_K(V^\perp, W^\perp) \mathrm{vol}(M) \mathrm{vol}(N). \tag{3.2}$$

It is clear that if G/K is a real space form then all $(\dim V)$-dimensional submanifolds are of type V. So (3.2) yields the Poincaré formula in real space forms.

Theorem 3.2. *Let G/K be a real space form of dimension n with $K \cong SO(n)$. Then for submanifolds M and N in G/K of dimensions p and q with $p + q \geq n$*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(SO(n+1)) \mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p) \mathrm{vol}(S^q)} \mathrm{vol}(M) \mathrm{vol}(N)$$

holds.

In the case where G/K is a complex space form, any p-dimensional complex submanifold is of type V for any p-dimensional complex subspace V in $T_o(G/K)$, and any Lagrangian submanifold is of type V for any Lagrangian subspace V in $T_o(G/K)$. Thus we have the following theorems.

Theorem 3.3. *Let G/K be a complex space form of complex dimension n with $K \cong U(1) \times U(n)$. Then for submanifolds M and N in G/K of complex dimensions p and q with $p + q \geq n$*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(U(n+1)) \mathrm{vol}(\mathbb{C}P^{p+q-n})}{\mathrm{vol}(\mathbb{C}P^p) \mathrm{vol}(\mathbb{C}P^q)} \mathrm{vol}(M) \mathrm{vol}(N)$$

holds.

Theorem 3.4. *Let G/K be a complex space form of complex dimension n with $K \cong U(1) \times U(n)$. Then for Lagrangian submanifolds M and N in G/K*

$$\int_G \#(M \cap gN) d\mu_G(g) = \frac{(n+1)\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{R}P^n)^2} \mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

3.2. **Kinematic formula in Riemannian homogeneous spaces.** For a linear subspace V of T, we define a vector space $\mathrm{II}(V)$ to be

$$\mathrm{II}(V) = \left\{ h \mid h : V \times V \to V^\perp ; \text{symmetric bilinear} \right\},$$

where $V^\perp$ is the normal space of V in T. A second fundamental form of a submanifold of G/K which passes through o and has V as the tangent space at o is an element of $\mathrm{II}(V)$. Let $K(V)$ be the stabilizer of V in K, that is, $K(V) = \{k \in K \mid k_* V = V\}$. The group $K(V)$ acts on $\mathrm{II}(V)$ in the following manner:

$$(kh)(u, v) = k_* \left(h(k_*^{-1} u, k_*^{-1} v) \right) \qquad (u, v \in V) \tag{3.3}$$

for $k \in K(V)$ and $h \in \mathrm{II}(V)$. Here we may consider a polynomial $\mathcal{P}$ on the vector space $\mathrm{II}(V)$ which is invariant under $K(V)$, that is, $\mathcal{P}(kh) = \mathcal{P}(h)$ for all $k \in K(V)$ and $h \in \mathrm{II}(V)$. In addition, let M be a submanifold of G/K of type V. For the second fundamental form h_x^M of M at $x \in M$, we define

$$\mathcal{P}(h_x^M) = \mathcal{P}(h_o^{g_x^{-1} M}).$$

Then we can define an integral invariant $I^{\mathcal{P}}(M)$ of M from a polynomial $\mathcal{P}$ by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h_x^M) d\mu_M(x). \tag{3.4}$$

We also define a vector space $\mathrm{EII}(T)$ to be

$$\mathrm{EII}(T) = \{ h \mid h : T \times T \to T ; \text{symmetric bilinear} \}.$$

Since K also acts on $\mathrm{EII}(T)$ in the same way with (3.3), we can define integral invariants from polynomials on $\mathrm{EII}(T)$ invariant under K in the same manner with (3.4).

With these preliminaries, we can now state the kinematic formulae in Riemannian homogeneous spaces as follows:

Theorem 3.5. ([5] paragraph 4.10) *Let V and W be linear subspaces of T with* $\dim(V) + \dim(W) \geq \dim(T)$*, and let $\mathcal{P}$ be a homogeneous polynomial of degree l on* $\mathrm{EII}(T)$ *which is invariant under K such that*

$$\int_K \sigma(V^\perp, k_* W^\perp)^{1-l} d\mu_K(k) < \infty. \tag{3.5}$$

Then there exists a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that

1. *each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(V)$ *invariant under $K(V)$,*
2. *each $\mathcal{R}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(W)$ *invariant under $K(W)$,*
3. $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ *for each α,*
4. *for any compact submanifolds (possibly with boundaries) M of type V and N of type W in G/K the kinematic formula*

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N) \tag{3.6}$$

holds.

The inequality (3.5) is a condition for the convergence of the integration. If G/K is a real space form, then the condition (3.5) can be replaced by the manageable inequality $l \le \dim(M) + \dim(N) - \dim(G/K) + 1$.

3.3. **Transfer principle.** Under the hypothesis of Theorem 3.5 let G' be a unimodular Lie group with $\dim G' = \dim G$ and K' be a compact subgroup of G' with isomorphism $\rho : K \to K'$ and $\mathrm{vol}(K) = \mathrm{vol}(K')$. We assume that there is a linear isometry $\psi : T \to T'$ such that

$$\psi \circ k_* = \rho(k)_* \circ \psi \qquad ({}^\forall k \in K).$$

Then ψ induces following isomorphisms of rings:

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{polynomials on EII}(T) \\ \text{invariant under } K \end{array} \right\} & \cong & \left\{ \begin{array}{l} \text{polynomials on EII}(T') \\ \text{invariant under } K' \end{array} \right\} \\ \mathcal{P} & \longmapsto & \mathcal{P}' \end{array}$$

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{polynomials on II}(V) \\ \text{invariant under } K(V) \end{array} \right\} & \cong & \left\{ \begin{array}{l} \text{polynomials on II}(\psi V) \\ \text{invariant under } K'(\psi V) \end{array} \right\} \\ \mathcal{Q}_\alpha,\ \mathcal{R}_\alpha & \longmapsto & \mathcal{Q}'_\alpha,\ \mathcal{R}'_\alpha \end{array}$$

Here we denote by $\mathcal{P}', \mathcal{Q}'_\alpha, \mathcal{R}'_\alpha$ the image of $\mathcal{P}, \mathcal{Q}_\alpha, \mathcal{R}_\alpha$ respectively under these isomorphisms. With this notation, if a kinematic formula (3.6) holds in G/K, then a kinematic formula

$$\int_{G'} I^{\mathcal{P}'}(M' \cap gN') d\mu_{G'}(g) = \sum_\alpha I^{\mathcal{Q}'_\alpha}(M') I^{\mathcal{R}'_\alpha}(N')$$

holds for any compact submanifolds M' of type ψV and N' of type ψW in G'/K'.

4. Some explicit expressions of kinematic formulae

In this section, we shall give some explicit forms of kinematic formulae stated in Theorem 3.5.

Take an orthonormal basis $e_1, \dots, e_n$ of T such that $e_1, \dots, e_p$ is a basis of V and $e_{p+1}, \dots, e_n$ is a basis of $V^\perp$. Then the components of $h \in$ II(V) and $H \in$ EII(T) are represented by

$$\begin{aligned} h^k_{ij} &= \langle h(e_i, e_j), e_k \rangle \qquad (1 \le i, j \le p,\ p+1 \le k \le n) \\ H^k_{ij} &= \langle H(e_i, e_j), e_k \rangle \qquad (1 \le i, j, k \le n) \end{aligned}$$

The following polynomials $\mathcal{W}_{2l}$ are homogeneous polynomials on II(V) of degree $2l$ invariant under $O(V) \times O(V^\perp)$.

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \le i_1, \dots, i_{2l} \le p \\ p+1 \le k_1, \dots, k_l \le n}} \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \cdots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \cdots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \cdots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \cdots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}.$$

We define homogeneous polynomials, also denoted by $\mathcal{W}_{2l}$, by

$$\mathcal{W}_{2l}(H) = \sum_{\substack{1\le i_1,\dots,i_{2l}\le n\\ 1\le k_1,\dots,k_l\le n}} \det \begin{bmatrix} H^{k_1}_{i_1 i_1} & H^{k_1}_{i_1 i_2} & \cdots & H^{k_1}_{i_1 i_{2l}} \\ H^{k_1}_{i_2 i_1} & H^{k_1}_{i_2 i_2} & \cdots & H^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ H^{k_l}_{i_{2l-1} i_1} & H^{k_l}_{i_{2l-1} i_2} & \cdots & H^{k_l}_{i_{2l-1} i_{2l}} \\ H^{k_l}_{i_{2l} i_1} & H^{k_l}_{i_{2l} i_2} & \cdots & H^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}$$

on $\mathrm{EII}(T)$ of degree $2l$ invariant under $O(T)$. In the both cases, $\mathcal{W}_0 = 1$ by definitions. A second fundamental form $h \in \mathrm{II}(V)$ can be extended to $H \in \mathrm{EII(T)}$ by

$$H(u, v) = h(Pu, Pv) \qquad (u, v \in T),$$

where $P : T \to V$ is the orthogonal projection. If $H \in \mathrm{EII(T)}$ is the extension of $h \in \mathrm{II}(V)$, then we have

$$\mathcal{W}_{2l}(h) = \mathcal{W}_{2l}(H).$$

Furthermore these polynomials $\mathcal{W}_{2l}$ are characterized as the invariant polynomials which vanish on the (extended) second fundamental forms with relative rank less than $2l$. For a submanifold M of G/K, we introduce the integral invariants $\mu_{2l}(M)$ defined by

$$\mu_{2l}(M) = I^{\mathcal{W}_{2l}}(M).$$

For these integral invariants μ_{2l}, we have the following remarkable formula.

Theorem 4.1 (Chern-Federer formula)**.** *Let G/K be a real space form of dimension n with $K \cong SO(n)$. Assume that $0 \le 2l \le p+q-n$. Then there exist constants $c(p, q, n, i, l)$ determined by indicated parameters so that*

$$\int_G \mu_{2l}(M \cap gN)d\mu_G(g) = \sum_{i=0}^{l} c(p, q, n, i, l)\mu_{2i}(M)\mu_{2(l-i)}(N)$$

holds for any compact submanifolds M and N in G/K of dimensions p and q, respectively.

The value of the constants $a(p, q, n, i, l)$ were computed by Chern [3], Nijenhuis [10].

The space of homogeneous polynomials on $\mathrm{II}(V)$ of degree 2 invariant under $O(V) \times O(V^\perp)$ is spanned by two polynomials

$$\mathcal{Q}_1(h) = \sum_{i,j,k} \left(h^k_{ij}\right)^2, \qquad \mathcal{Q}_2(h) = \sum_k \left(\sum_i h^k_{ii}\right)^2,$$

where $1 \le i, j \le p,\ p+1 \le k \le n$. And if $2 \le p \le n-1$ these two polynomials are independent. Geometrically, $\mathcal{Q}_1(h)$ is the square of the norm of the second fundamental form, and $\mathcal{Q}_2(h)$ is p^2 times the square of the mean curvature. However, it is convenient for us to take the basis

$$\mathcal{W}_2 = \mathcal{Q}_2 - \mathcal{Q}_1, \qquad \mathcal{U}_p = p\mathcal{Q}_1 - \mathcal{Q}_2.$$

For these polynomials we have the following:

Theorem 4.2. *Let G/K be a real space form of dimension n with $K \cong SO(n)$. Assume that $2 \le p+q-n$. Then there exist constants $a(p, q, n)$ and $b(p, q, n)$ so that*

$$\int_G I^{\mathcal{W}_2}(M \cap gN)d\mu_G(g) = a(p, q, n)I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + a(q, p, n)\mathrm{vol}(M)I^{\mathcal{W}_2}(N)$$

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN)d\mu_G(g) = b(p, q, n)I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + b(q, p, n)\mathrm{vol}(M)I^{\mathcal{U}_q}(N)$$

holds for any compact submanifolds M and N in G/K of dimensions p and q, respectively.

The first one is entirely the Chern-Federer formula of degree 2. The second one was suggested by Howard [5]. Finally Kang-Suh-S. [7] gave the explicit forms completely.

The polynomial $\mathcal{U}_p$ is characterized as the invariant polynomial which vanishes at an umbilic point. The integral invariant

$$I^{\mathcal{U}_p^{p/2}}(M) = \int_M \left(\mathcal{U}_p(h_x^M)\right)^{p/2} d\mu_M(x)$$

is an conformal invariant, called the Willmore-Chen functional, of p dimensional submanifold M (see [1], [2], [17]).

5. Transferred kinematic formulae in rank one symmetric spaces

By the transfer principle we can obtain some kinematic formulae in rank one symmetric spaces transferring from the case of real space forms.

Let G/K be a rank one symmetric space, that is one of a sphere $S^n = SO(n+1)/SO(n)$, a real projective space $\mathbb{R}P^n = SO(n+1)/S(O(1) \times O(n))$, a complex projective space $\mathbb{C}P^n = U(n+1)/U(1) \times U(n)$, a quaternionic projective space $\mathbb{H}P^n = Sp(n+1)/Sp(1) \times Sp(n)$, the Cayley projective plane $\mathbf{Cay}P^2 = F_4/Spin(9)$, and their non-compact duals. Then G/K is isotropic, that is, K acts transitively on the sphere in T by the linear isotropy representation. Therefore, for $v \in T$ we denote by $K(v)$ the stabilizer of v in K, then $K/K(v)$ is homothetic to the unit sphere. It is clear that any real hypersurfaces in a rank one symmetric space is submanifolds of type V for any hyperplane V in T.

Theorem 5.1. ([5] paragraph 3.12) *Let G/K be a rank one symmetric space of dimension n. Let M be a submanifold of dimension p and N a real hypersurface in G/K. Then*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}\mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

When G/K is a rank one symmetric space except S^n, $\mathbb{R}P^n$ and $\mathbb{R}H^n$, K does not act transitively on $Gr_p(T)$ for $p \geq 2$. Therefore Theorem 5.1 is outside of Theorem 3.5. Howard showed this formula in a geometric way like the transfer principle. Here we prove this in a different way using generalized Poincaré formula.

Proof. Let $V \in Gr_{p-1}(T)$ and $v \in T$. Set $W = \mathbb{R}v$. Then

$$\begin{aligned} \sigma_K(V,W) &= \int_K \sigma(V,\ k_*W) d\mu_K(k) \\ &= \mathrm{vol}(K(v)) \int_{K/K(v)} \sigma(V,\ [k]_*W) d\mu_{K/K(v)}([k]) \\ &= \frac{\mathrm{vol}(K)}{\mathrm{vol}(S^{n-1})} \int_{K/K(v)} \sigma(V,\ [k]_*W) d\mu_{S^{n-1}}([k]). \end{aligned}$$

In the second equality the integration on K is reduced to that on $K/K(v)$. The last equality means that the invariant measure on $K/K(v)$ is normalized to be the unit sphere S^{n-1}. The last integration on the unit sphere does not depend on V, and is equal for any rank one symmetric spaces. Therefore we have

$$\sigma_K(V,W) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))}\sigma_{SO(n)}(V,W).$$

Form Theorem 3.2 we have

$$\sigma_{SO(n)}(V, W) = \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p-1})}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}.$$

Thus we obtain the theorem. □

Poincaré formula is a kinematic formula for the integral invariant of degree 0, that is volume functional. A fundamental question is how to express kinematic formulae for other integral invariants in rank one symmetric spaces. From Theorem 4.2 we now have the following formulae.

Theorem 5.2 ([15])**.** *Let M and N be real hypersurfaces in a rank one symmetric space G/K of dimension n. Then the following kinematic formulae hold:*

$$\begin{aligned}
&\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) \\
&\quad = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1, n-1, n)\big(I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{W}_2}(N)\big), \\
&\int_G I^{\mathcal{U}_{n-2}}(M \cap gN) d\mu_G(g) \\
&\quad = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n-1, n-1, n)\big(I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{U}_{n-1}}(N)\big).
\end{aligned}$$

6. Applications of kinematic formulae

6.1. **Hadwiger's containment problem.** In 1941, Hadwiger obtained some sufficient conditions for one domain to contain another domain in $\mathbb{R}^2$ (see [16]). Later Zhou generalized his results to higher dimensions and other real space forms using some famous kinematic formulae. Here we shall work in the case of $\mathbb{R}^3$ for simplicity, although he obtained some results in the more general situations ([20], [21]). The idea of the proof is essentially same.

In this subsection, we use the following notation. For domains D_i $(i = 0, 1)$ in $\mathbb{R}^3$, we denote their volumes by V_i, and the Euler-Poincaré characteristic by $\chi(\cdot)$. For surfaces M_i or ∂D_i $(i = 0, 1)$ in $\mathbb{R}^3$, we denote the surface areas, the total Gaussian curvatures, total mean curvatures and total square mean curvatures by $F_i, \tilde{K}_i, \tilde{H}_i$ and $\tilde{H}_i^{(2)}$, respectively.

Theorem 6.1 (Chern)**.** *Let D_0 and D_1 be two domains bounded by simple surfaces ∂D_0 and ∂D_1 in $\mathbb{R}^3$. Then*

$$\int_{I(\mathbb{R}^3)} \chi(D_0 \cap gD_1) d\mu(g) = 8\pi^2(V_0\chi(D_1) + \chi(D_0)V_1) + 2\pi(F_0\tilde{H}_1 + \tilde{H}_0 F_1)$$

holds.

Theorem 6.2 (C.S. Chen)**.** *Let M_0 and M_1 be two surfaces in $\mathbb{R}^3$. Then*

$$\int_{I(\mathbb{R}^3)} \left(\int_{M_0 \cap gM_1} \kappa_g^2 ds \right) d\mu(g) = 2\pi^3(3\tilde{H}_0^{(2)} - \tilde{K}_0)F_1 + 2\pi^3 F_0(3\tilde{H}_1^{(2)} - \tilde{K}_1)$$

holds, where κ_g denotes the curvature of the intersection curve $M_0 \cap gM_1$.

Theorem 6.3 (Zhou [19]). *Let D_0 and D_1 be connected domains bounded by simple surfaces ∂D_0 and ∂D_1 in $\mathbb{R}^3$. Assume that there exists a finite integer N_0 such that $\chi(D_0 \cap gD_1) \leq N_0$ for all $g \in I(\mathbb{R}^3)$. If*

$$\begin{aligned}&8\pi(V_0\chi(D_1) + \chi(D_0)V_1) + 2(F_0\tilde{H}_1 + \tilde{H}_0F_1)\\&-N_0\pi\Big(F_0F_1\big((3\tilde{H}_0^{(2)} - \tilde{K}_0)F_1 + F_0(3\tilde{H}_1^{(2)} - \tilde{K}_1)\big)\Big)^{\frac{1}{2}} > 0,\end{aligned}$$

then there exists $g \in I(\mathbb{R}^3)$ so that $gD_0 \subset D_1$ or $gD_1 \subset D_0$.

Proof.

$$\begin{aligned}&\mathrm{vol}\{g \in I(\mathbb{R}^3) \mid gD_1 \subset D_0 \text{ or } gD_0 \subset D_1\}\\&= \mathrm{vol}\{g \in I(\mathbb{R}^3) \mid D_0 \cap gD_1 \neq \emptyset\} - \mathrm{vol}\{g \in I(\mathbb{R}^3) \mid \partial D_0 \cap g\partial D_1 \neq \emptyset\}\end{aligned}$$

By Chern's kinematic formula, we can estimate the first term of the right hand side from below.

$$\begin{aligned}N_0\int_{\{g\in I(\mathbb{R}^3)\ \mid\ D_0\cap gD_1\neq\emptyset\}} d\mu(g) &\geq \int_{I(\mathbb{R}^3)} \chi(D_0 \cap gD_1)d\mu(g)\\&= 8\pi^2(V_0\chi(D_1) + \chi(D_0)V_1) + 2\pi(F_0\tilde{H}_1 + \tilde{H}_0F_1).\end{aligned}$$

By Fenchel's theorem, the Poincaré formula and Chen's kinematic formula, we can estimate the second term from above.

$$\begin{aligned}&2\pi\int_{\{g\in I(\mathbb{R}^3)\ \mid\ \partial D_0\cap g\partial D_1\neq\emptyset\}} d\mu(g)\\&\leq \int_{I(\mathbb{R}^3)}\left(\int_{\partial D_0\cap g\partial D_1}\kappa_g ds\right)d\mu(g)\\&\leq \int_{I(\mathbb{R}^3)}\left(\int_{\partial D_0\cap g\partial D_1}1^2 ds\right)^{\frac{1}{2}}\left(\int_{\partial D_0\cap g\partial D_1}\kappa_g^2 ds\right)^{\frac{1}{2}}d\mu(g)\\&\leq \left(\int_{I(\mathbb{R}^3)}\mathrm{Length}(\partial D_0\cap g\partial D_1)d\mu(g)\right)^{\frac{1}{2}}\left(\int_{I(\mathbb{R}^3)}\left(\int_{\partial D_0\cap g\partial D_1}\kappa_g^2 ds\right)d\mu(g)\right)^{\frac{1}{2}}\\&= \big(2\pi^3F_0F_1\big)^{\frac{1}{2}}\big(2\pi^3(3\tilde{H}_0^{(2)} - \tilde{K}_0)F_0 + 2\pi^3F_0(3\tilde{H}_1^{(2)} - \tilde{K}_1)\big)^{\frac{1}{2}}.\end{aligned}$$

Therefore we can estimate $\mathrm{vol}\{g \in I(\mathbb{R}^3) \mid gD_1 \subset D_0 \text{ or } gD_0 \subset D_1\}$ from below. If $\mathrm{vol}\{g \in I(\mathbb{R}^3) \mid gD_1 \subset D_0 \text{ or } gD_0 \subset D_1\} > 0$, then there does exist $g \in I(\mathbb{R}^3)$ such that $gD_0 \subset D_1$ or $gD_1 \subset D_0$. Thus we obtain the theorem. □

6.2. **Homologically volume minimizing submanifolds.** In general, the Poincaré formulae, that is kinematic formulae for volume functional, can not be expressed as a constant times of the volumes of two submanifolds. However, in some cases we can estimate the evaluation of the integral. Then, by that inequality, we can prove the homologically volume minimizing property of certain submanifolds.

Let $G_r(\mathbb{R}^{r+n})$ denote the real Grassmannian manifold of all r-dimensional subspaces in $\mathbb{R}^{r+n}$. $SO(r+n)$ acts on $G_r(\mathbb{R}^{r+n})$ transitively and it can be expressed as a homogeneous space

$$G_r(\mathbb{R}^{r+n}) \cong SO(r+n)/S(O(r) \times O(n)).$$

Moreover $G_r(\mathbb{R}^{r+n})$ has an invariant Riemannian metric as a symmetric space. In her paper [8], Lê Hông Vân obtained the following inequality.

Theorem 6.4 (Lê Hông Vân [8]). *Let N be an rs-dimensional submanifold of $G_r(\mathbb{R}^{r+n})$. Then*

$$\int_{SO(r+n)} \#(N \cap gG_r(\mathbb{R}^{r+n-s}))d\mu_{SO(r+n)}(g) \leq \frac{\mathrm{vol}(SO(r+n))}{\mathrm{vol}(G_r(\mathbb{R}^{r+s}))}\mathrm{vol}(N).$$

By this inequality, we can prove the following.

Theorem 6.5 (Lê Hông Vân [8]). *$G_r(\mathbb{R}^{r+s})$ is a volume minimizing submanifold of $G_r(\mathbb{R}^{r+n})$ in its $\mathbb{Z}_2$-homology class.*

Proof. Let N be a compact submanifold of $G_r(\mathbb{R}^{r+n})$ in the $\mathbb{Z}_2$-homology class of $G_r(\mathbb{R}^{r+s})$. Since $\#(G_r(\mathbb{R}^{r+s}) \cap gG_r(\mathbb{R}^{r+n-s})) = 1$ for almost all $g \in SO(r+n)$, we have

$$\#(N \cap gG_r(\mathbb{R}^{r+n-s})) \equiv 1 \pmod 2$$

for almost all $g \in SO(r+n)$. Thus from Theorem 6.4 we have

$$\begin{aligned}\mathrm{vol}(SO(r+n)) &\leq \int_{SO(r+n)} \#(N \cap gG_r(\mathbb{R}^{r+n-s}))d\mu(g) \\ &= \frac{\mathrm{vol}(SO(r+n))}{\mathrm{vol}(G_r(\mathbb{R}^{r+s}))}\mathrm{vol}(N)\end{aligned}$$

Hence we have

$$\mathrm{vol}(G_r(\mathbb{R}^{r+s})) \leq \mathrm{vol}(N).$$

□

Furthermore, she proved homologically volume minimizing properties of sub-Grassmannians of complex and quaternionic Grassmannian manifolds.

Theorem 6.6 (Lê Hông Vân [8]). *$G_r(\mathbb{C}^{r+s})$ is a volume minimizing submanifold of $G_r(\mathbb{C}^{r+n})$ in its $\mathbb{Z}_2$-homology class.*

Theorem 6.7 (Lê Hông Vân [8]). *$G_r(\mathbb{H}^{r+s})$ is a volume minimizing submanifold of $G_r(\mathbb{H}^{r+n})$ in its $\mathbb{Z}_2$-homology class.*

By the same method, Liu [9] proved that the Pontryagin cycle in $SO(n)$ is homologically volume minimizing in its homology class. And Tasaki [18] proved that the Helgasson spheres in some symmetric spaces are homologically volume minimizing in their homology classes.

6.3. **Hamiltonian volume minimizing properties of Lagrangian submanifolds.** We apply the Poincaré formula to show Hamiltonian volume minimizing properties of minimal Lagrangian submanifolds in some Kähler manifolds.

Let (M, ω) be a $2n$-dimensional closed symplectic manifold with symplectic form ω and L be an n-dimensional closed submanifold of M. Then L is said to be *Lagrangian* if $\omega|_{TL} \equiv 0$. Hamiltonian isotopies of (M, ω) are defined as follows. If a smooth function $F : M \times [0, 1] \to \mathbb{R}$ is given, then we can uniquely define the vector field X_t on M for each $t \in [0, 1]$ such that

$$\omega(X_t, \cdot) = dF(\cdot, t).$$

Therefore, we have the flow $\{\phi_t\}_{t\in[0,1]}$ of diffeomorphisms on M defined by the differential equation

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x))$$

with initial condition $\phi_0 = id_M$. The time 1-map ϕ_1 of this flow is called a *Hamiltonian diffeomorphism*. The set of all Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

Hereafter we restrict our attention to Kähler manifolds to introduce the volume functional. Let (M, ω, J) be a closed connected Kähler manifold. A Lagrangian submanifold L in M is said to be *Hamiltonian volume minimizing*, if L has the least among all Hamiltonian deformations of L. This notion was first introduced by Y. G. Oh [11], and he mentioned only one non-trivial example:

Theorem 6.8 (Kleiner-Oh)**.** *The real projective space* $\mathbb{R}P^n$ *canonically embedded in* $\mathbb{C}P^n$ *is a Hamiltonian volume minimizing Lagrangian submanifold.*

In order to prove this theorem, we use the Poincaré formula for Lagrangian submanifolds in $\mathbb{C}P^n$ (Theorem 3.4), and the following Arnold-Givental inequality in the Lagrangian intersection theory.

Theorem 6.9 (Oh [12], [13] and [14])**.** *Let* (M, ω) *be a compact symplectic manifold such that there exists an integrable almost complex structure* J *for which the triple* (M, ω, J) *becomes a compact Hermitian symmetric space. Let* $L = \mathrm{Fix}\,\tau$ *be the fixed point set of an anti-holomorphic involutive isometry* τ *on* M*. Assume that the minimal Maslov number of* L *is greater than or equal to* 2*. Then for any Hamiltonian diffeomorphism* ϕ *of* M *such that* L *and* $\phi(L)$ *intersect transversely, the inequality*

$$\sharp(L \cap \phi(L)) \geq SB(L, \mathbb{Z}_2) := \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) \tag{6.1}$$

holds.

Proof of Theorem 6.8. Let L be the real projective space $\mathbb{R}P^n$ canonically embedded in $\mathbb{C}P^n$. From Theorems 6.9 and 3.4, for any $\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$ we have

$$\begin{aligned}
\frac{(n+1)\mathrm{vol}(U(n+1))}{(\mathbb{R}P^n)^2}\mathrm{vol}(L)\mathrm{vol}(\phi(L)) &\geq \int_{U(n+1)} \sharp(L \cap g \circ \phi(L)) d\mu_{U(n+1)}(g) \\
&\geq \int_{U(n+1)} SB(L, \mathbb{Z}_2) d\mu_{U(n+1)}(g) \\
&= \mathrm{vol}(U(n+1)) SB(L, \mathbb{Z}_2) \\
&= (n+1)\mathrm{vol}(U(n+1)).
\end{aligned}$$

Hence we have

$$\mathrm{vol}(\phi(L)) \geq \mathrm{vol}(L).$$

□

By the same method, we can prove the following theorem.

Theorem 6.10 (Iriyeh-Ono-S. [6])**.** *A totally geodesic Lagrangian torus* $S^1(1) \times S^1(1)$ *in* $S^2(1) \times S^2(1)$ *is Hamiltonian volume minimizing.*

In this case, however Poincaré formula can not express as a constant times of the volumes of two Lagrangian submanifolds, we can estimate the integral of intersection numbers.

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