Introduction to integral geometry in Riemannian homogeneous spaces, III

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Examples

Theorem (Poincaré formula)

$M^p, N^q \subset G/K$: *real space form*

$$\int_G \mathrm{vol}(M \cap gN) d\mu(g) = C(p, q, n)\mathrm{vol}(M)\mathrm{vol}(N)$$

Theorem

$M^p, N^q \subset G/K$: *a real space form*

$0 \leq 2l \leq p + q - n$

$$\int_G \mu_{2l}(M \cap gN) d\mu_G(g) = \sum_{i=0}^{l} c(p, q, n, i, l)\mu_{2i}(M)\mu_{2(l-i)}(N)$$

Integral invariants

G/K : Riem. homog. sp.
$V \subset T_o(G/K)$: subspace

$$\begin{aligned} \mathrm{II}(V) &= \left\{ h : V \times V \to V^{\perp}; \text{symmetric bilinear} \right\} \\ K(V) &= \{ k \in K \mid k_* V = V \} \end{aligned}$$

$K(V)$ acts on $\mathrm{II}(V)$ by

$$(kh)(u,v) = k_* h(k_*^{-1} u, k_*^{-1} v) \quad (u, v \in V)$$

$\mathcal{P}$: polynomial on $\mathrm{II}(V)$ invariant under $K(V)$
$M \subset G/K$: submanifold of type V

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M$$

Kinematic formula (Howard)

G/K : Riem. homog. space, G : unimodular

$V, W \subset T$, $\dim V + \dim W \geq \dim T$

$\mathcal{P}$: K-inv. homog. poly. on $\mathrm{EII}(T)$

Then there exist finite pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ s.t.

(1) $\mathcal{Q}_\alpha$: $K(V)$-inv. homog. poly. on $\mathrm{II}(V)$

(2) $\mathcal{R}_\alpha$: $K(W)$-inv. homog. poly. on $\mathrm{II}(W)$

(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = \deg \mathcal{P}$ for each α

(4) for any submanifolds M of type V and N of type W in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$$\rho \ : \ K \to K' \ ; \text{ isomorphism}$$

$$\psi \ : \ T_o(G/K) \to T_{o'}(G'/K') \ ; \text{ linear isometry s.t.}$$

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

$$\Rightarrow \left\{ \begin{array}{r} K(V)\text{-inv. poly.} \\ \text{on } \mathrm{II}(V) \end{array} \right\} \overset{\psi}{\cong} \left\{ \begin{array}{r} K'(\psi V)\text{-inv. poly.} \\ \text{on } \mathrm{II}(\psi V) \end{array} \right\}$$

$$\Rightarrow \int_{G'} I^{\mathcal{P}'}(M' \cap gN') d\mu_{G'}(g) = \sum_\alpha I^{\mathcal{Q}'_\alpha}(M') I^{\mathcal{R}'_\alpha}(N')$$

holds for M' of type ψV and N' of type ψW in G'/K'.

Case of real space forms

$e_1, \dots, e_n$: o.n.b. of $T_o(G/K)$ s.t.

$e_1, \dots, e_p$ is a basis of V and $e_{p+1}, \dots, e_n$ is a basis of $V^\perp$

Then the components of $h \in \mathrm{II}(V)$ are given by

$$h_{ij}^k = \langle h(e_i, e_j), e_k \rangle \qquad (1 \le i, j \le p,\ p+1 \le k \le n)$$

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \le i_1, \dots, i_{2l} \le p \\ p+1 \le k_1, \dots, k_l \le n}} \det \begin{bmatrix} h_{i_1 i_1}^{k_1} & h_{i_1 i_2}^{k_1} & \dots & h_{i_1 i_{2l}}^{k_1} \\ h_{i_2 i_1}^{k_1} & h_{i_2 i_2}^{k_1} & \dots & h_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_{2l-1} i_1}^{k_l} & h_{i_{2l-1} i_2}^{k_l} & \dots & h_{i_{2l-1} i_{2l}}^{k_l} \\ h_{i_{2l} i_1}^{k_l} & h_{i_{2l} i_2}^{k_l} & \dots & h_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}$$

is $K(V) = O(V) \times O(V^\perp)$-inv. homog. poly. on $\mathrm{II}(V)$ of degree $2l$

$$\mu_{2l}(M) := I^{\mathcal{W}_{2l}}(M).$$

Invariant polynomials of degree 2

The space of homogeneous polynomials of degree 2 on $\mathrm{II}(V)$ invariant under $O(V) \times O(V^{\perp})$ is spaned by

$$\mathcal{Q}_1(h) = \sum_{\substack{1\le i,j\le p\\ p+1\le k\le n}} \left(h^k_{ij}\right)^2 = \|h\|^2$$

$$\mathcal{Q}_2(h) = \sum_{p+1\le k\le n} \left(\sum_{1\le i\le p} h^k_{ii}\right)^2 = p^2 H^2$$

However, we shall take a basis as:

$$\mathcal{W}_2(h) = \mathcal{Q}_2(h) - \mathcal{Q}_1(h)$$

$$\mathcal{U}_p(h) = p\mathcal{Q}_1(h) - \mathcal{Q}_2(h)$$

Kinematic formulae of degree 2

Theorem (Chern-Federer, Howard, Kang-Suh-S.)

$M^p, N^q \subset G/K$: *a real space form*

Assume $2 \le p + q - n$

$$\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g)$$
$$= a(p,q,n) I^{\mathcal{W}_2}(M) \mathrm{vol}(N) + a(q,p,n) \mathrm{vol}(M) I^{\mathcal{W}_2}(N)$$
$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN) d\mu_G(g)$$
$$= b(p,q,n) I^{\mathcal{U}_p}(M) \mathrm{vol}(N) + b(q,p,n) \mathrm{vol}(M) I^{\mathcal{U}_q}(N)$$

Transferred kinematic formulae

Theorem (Howard)

G/K : *rank one symmetric space*

$M^p \subset G/K$: *submanifold,* $N^{n-1} \subset G/K$: *real hypersurface*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}\mathrm{vol}(M)\mathrm{vol}(N)$$

Transferred kinematic formulae

Theorem (S.)

G/K : rank one symmetric space $M^{n-1}, N^{n-1} \subset G/K$: real hypersurfaces

$$\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n) \big(I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{W}_2}(N) \big),$$

$$\int_G I^{\mathcal{U}_{n-2}}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n) \big(I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{U}_{n-1}}(N) \big).$$

Hadwiger's containment problem

Theorem (Zhou)

Let D_0 and D_1 be connected domains bounded by simple surfaces ∂D_0 and ∂D_1 in $\mathbb{R}^3$. Assume that there exists a finite integer N_0 such that $\chi(D_0 \cap gD_1) \leq N_0$ for all $g \in I(\mathbb{R}^3)$. If

$$8\pi(V_0\chi(D_1) + \chi(D_0)V_1) + 2(F_0\tilde{H}_1 + \tilde{H}_0F_1)$$
$$-N_0\pi\Big(F_0F_1((3\tilde{H}_0^{(2)} - \tilde{K}_0)F_1 + F_0(3\tilde{H}_1^{(2)} - \tilde{K}_1))\Big)^{\frac{1}{2}} > 0,$$

then there exists $g \in I(\mathbb{R}^3)$ so that $gD_0 \subset D_1$ or $gD_1 \subset D_0$.

Theorem (Chern)

Let D_0 and D_1 be two domains bounded by simple surfaces ∂D_0 and ∂D_1 in $\mathbb{R}^3$. Then

$$\int_{I(\mathbb{R}^3)} \chi(D_0 \cap gD_1) d\mu(g)$$
$$= 8\pi^2 (V_0 \chi(D_1) + \chi(D_0) V_1) + 2\pi (F_0 \tilde{H}_1 + \tilde{H}_0 F_1)$$

Theorem (C.S. Chen)

Let M_0 and M_1 be two surfaces in $\mathbb{R}^3$. Then

$$\int_{I(\mathbb{R}^3)} \left(\int_{M_0 \cap gM_1} \kappa_g^2 ds \right) d\mu(g)$$
$$= 2\pi^3 (3\tilde{H}_0^{(2)} - \tilde{K}_0) F_1 + 2\pi^3 F_0 (3\tilde{H}_1^{(2)} - \tilde{K}_1)$$

holds, where κ_g denotes the curvature of $M_0 \cap gM_1$.

Homologically volume minimizing submanifolds

Theorem (Lê Hông Vân)

Let N be an rs-dimensional submanifold of $G_r(\mathbb{R}^{r+n})$. Then

$$\int_{SO(r+n)} \#(N \cap gG_r(\mathbb{R}^{r+n-s}))d\mu(g) \leq \frac{\mathrm{vol}(SO(r+n))}{\mathrm{vol}(G_r(\mathbb{R}^{r+s}))}\mathrm{vol}(N).$$

Theorem (Lê Hông Vân)

$G_r(\mathbb{R}^{r+s})$ is a volume minimizing submanifold of $G_r(\mathbb{R}^{r+n})$ in its $\mathbb{Z}_2$-homology class.