Weakly reflective orbits and tangentially degenerate orbits of s-representations

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Definition of a reflective submanifold

Definition (D. Leung)

Let $\widetilde{M}$ be a complete Riemannian manifold. A connected component M of the fixed point set of an involutive isometry σ of $\widetilde{M}$ is called a **reflective submanifold**.

- A reflective submanifold is a complete totally geodesic submanifold.
- For all $x \in M$ and all $\xi \in T_x^{\perp} M$, the reflection σ satisfies

$$\sigma(x) = x, \qquad (d\sigma)_x \xi = -\xi, \qquad \sigma(M) = M.$$

Definition of a weakly reflective submanifold

Definition

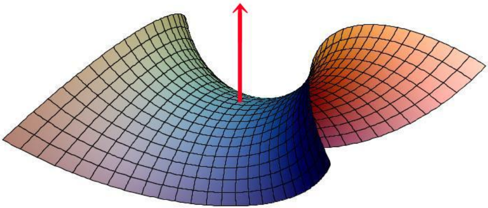
$M \subset \widetilde{M}$: **weakly reflective submanifold (WRS)**

$\overset{\text{def}}{\iff}$ for each $x \in M$ and each $\xi \in T_x^{\perp} M$,

there exists an isometry σ_ξ of $\widetilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M.$$

We call σ_ξ a **reflection** of M with respect to ξ.



An example of a weakly reflective submanifold

Example

$$M = S^{n-1}(1) \times S^{n-1}(1) = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid u, v \in S^{n-1}(1)\}$$

is a weakly reflective submanifold in $S^{2n-1}(\sqrt{2})$.

$x = (e_1, e_1) \in M$
$\xi = (e_1, -e_1) \in T_x^{\perp} M$

$$\begin{array}{rcl} \sigma_\xi : S^{2n-1}(\sqrt{2}) & \longrightarrow & S^{2n-1}(\sqrt{2}) \\ (u,v) & \longmapsto & (v,u) \end{array}$$

Then σ_ξ satisfies

$$\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(M) = M$$

Definition of an austere submanifold

Definition (Harvey-Lawson)

$M \subset \widetilde{M}$: **austere submanifold**
$\overset{\text{def}}{\Longleftrightarrow}$ for all $\xi \in T_x^{\perp} M$, the set of eigenvalues of the shape operator A_ξ of M is invariant under the multiplication by -1, concerning multiplicities.

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

Twisted normal bundle (Harvey-Lawson, Borrelli-Gorodski)

$M \subset S^n$: submanifold

$$\begin{array}{rcl} \Phi : N^1M \times S^1 & \longrightarrow & S^{2n+1} \subset \mathbf{R}^{2n+2} \\ (v_x,\ e^{i\theta}) & \longmapsto & (\cos\theta\, x,\ \sin\theta\, v_x) \end{array}$$

- A_ξ does not have 0-eigenvalue $\Longrightarrow$ Φ : Legendrian immersion
- $M \subset S^n$: austere $\Longrightarrow$ Φ : minimal

Classes of submanifolds

Proposition

$$\text{reflective} \subset \text{WRS} \subset \text{austere} \subset \text{minimal}$$

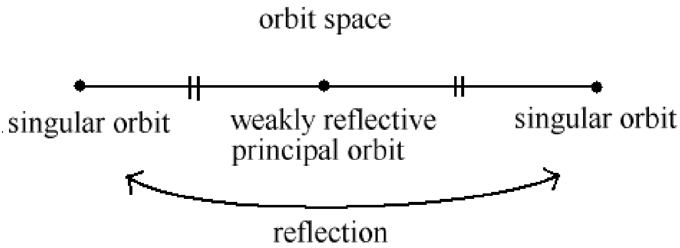
Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeniety one action on a Riemannian manifold is a weakly reflective submanifold.

Orbits of cohomogeneity one actions

Proposition

Let G be a connected Lie group acting isometrically on a complete, connected Riemannian manifold $\tilde{M}$. Suppose that the action of G on $\tilde{M}$ is cohomogeneity one with two singular orbits. If there exists a principal orbit which is a weakly reflective submanifold in $\tilde{M}$, then it has a same distance from two singular orbits and two singular orbits are isometric.



Orbits of s-representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

The linear isotropy representation $K \curvearrowright \mathfrak{m}$ is called an *s-representation*. For $H \in S \subset \mathfrak{m}$, the orbit $\mathrm{Ad}(K)H \subset S$ through H is called an *R-space*.

Related topics

- an orbit of the s-representation of a symmetric space of rank 2 $\Longleftrightarrow$ a homogeneous hypersurface in S^n
- G : compact, simple Lie algebra.
  $\mathrm{Ad}(G)H \subset \mathfrak{g} \Longleftrightarrow$ a Kähler C-space
- a symmetric R-space $\Longleftrightarrow$ a symmetric submanifold in $\mathbb{R}^n$
- an s-representation $\Longleftrightarrow$ a polar representation

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$\mathfrak{a} \subset \mathfrak{m}$: maximal abelian subspace

$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ ({}^\forall H \in \mathfrak{a})\}$$

$R = \{\lambda \in \mathfrak{a} \mid \mathfrak{m}_\lambda \neq \{0\}\}$: restricted root system

F : fundamental system of R

$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle > 0 \ (\forall \alpha \in F)\} \quad \text{: Weyl chamber}$$

$$\mathrm{Ad}(K)\bar{C} = \mathfrak{m} \quad \Longleftrightarrow \quad \bar{C} \cap S \quad \text{: orbit space}$$

For $\Delta \subset F$

$$C^\Delta = \{H \in \bar{C} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in \Delta),\ \langle \beta, H\rangle = 0 \ (\beta \in F - \Delta)\}$$

$$\bar{C} = \bigcup_{\Delta \subset F} C^\Delta \quad \text{disjoint union}$$

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Orbits of s-representations

Assume $H \in C^{\Delta} \subset \bar{C}$

$$\begin{aligned} Z_K^H &= \{k \in K \mid \mathrm{Ad}(K)H = H\} \\ &= \{k \in K \mid \mathrm{Ad}(K)\big|_{C^{\Delta}} = \mathrm{id}_{C^{\Delta}}\} \end{aligned}$$

$$\mathrm{Ad}(K)H \cong K/Z_K^H$$

$$\Delta_1 \subset \Delta_2 \subset F \quad \Longleftrightarrow \quad C^{\Delta_1} \subset \bar{C}^{\Delta_2}$$

$$H_1 \in C^{\Delta_1}, H_2 \in C^{\Delta_2} \quad \Longrightarrow \quad Z_K^{H_1} \supset Z_K^{H_2}$$

Theorem (Hirohashi-Song-Takagi-Tasaki)

For a subset $\Delta \subset F$ there exists unique $H \in C^{\Delta} \cap S$ such that $\mathrm{Ad}(K)H$ is a minimal submanifold in the hypersphere S.

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Austere orbits of s-representations

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is an austere submanifold in the hypersphere is one of the following list:

1. An orbit through a restricted root
2. $R = A_2; \quad H = 2e_1 - e_2 - e_3,\ e_1 + e_2 - 2e_3$
3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$
4. $R = D_n; \quad H = e_1$
5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$
6. $R = B_2$ with constant multiplicities; $\quad H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$ (regular orbit)
7. $R = G_2; \quad H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}} \qquad$ (regular orbit)

Weakly reflective orbits of s-representations

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is a weakly reflective submanifold in the hypersphere is one of the following list:

1. An orbit through a restricted root
2. $R = A_2; \quad H = 2e_1 - e_2 - e_3,\ e_1 + e_2 - 2e_3$
3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$
4. $R = D_n; \quad H = e_1$
5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$

Gauss mapping

$f : M^l \longrightarrow S^n$ immersion

$$\text{Gauss map} \quad \gamma : M \longrightarrow G_{l+1}(\mathbb{R}^{n+1})$$
$$x \longmapsto \mathbb{R}f(x) \oplus T_{f(x)}(f(M))$$

Definition

A submanifold $f(M) \subset S^n$ is said to be **tangenetially degenerate** if the Gauss map γ is degenerate.

Ferus inequality

Theorem (Ferus)

M^l : compact, connected manifold, $f : M \longrightarrow S^n$: immersion

Then, $\operatorname{rank}\gamma < F(l) \implies \operatorname{rank}\gamma = 0$

$$F(l) = \min\{k \mid A(k) + k \geq l\} \ : \ \text{Ferus number}$$

$$A((2k+1)2^{c+4d}) = 2^c + 8d - 1 \ : \ \text{Adams number}$$

$$(0 \leq c \leq 3,\ 0 \leq d)$$

Problem (Ishikawa-Kimura-Miyaoka)

1. Is the inequality $\operatorname{rank}\gamma < F(l)$ best possible for the implication $\operatorname{rank}\gamma = 0$?
2. Classify tangentially degenerate immersions $f : M^l \longrightarrow S^n$ with $\operatorname{rank}\gamma = F(l)$.

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Tangentially degenerate orbits

Theorem

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is tangentially degenerate is one of the following list:

1. H : a long root
2. $R = G_2$; H : a short root

Moreover, if λ is such a root, then

$$\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda.$$

- These orbits are weakly reflective submanifolds.

In general, for the Gauss map γ of a submanifold $M \subset S^n$

$$\begin{aligned} \ker(d\gamma)_x &= \{X \in T_x(M) \mid h(X,Y) = 0, \ ^\forall Y \in T_x(M)\} \\ &= \bigcap_{\xi \in T_x^\perp(M)} \ker(A_\xi) \end{aligned}$$

For the Gauss map γ of the orbit $\mathrm{Ad}(K)H$

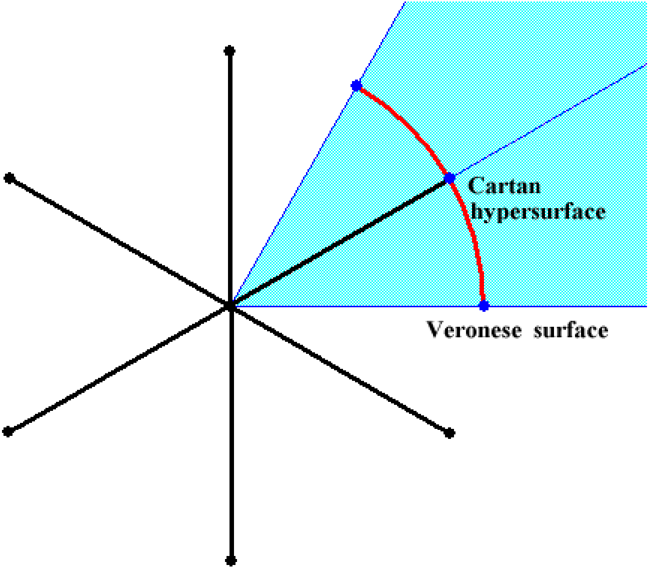
$$\ker(d\gamma)_H = \bigcap_{k \in (Z_K^H)_0} \mathrm{Ad}(k) \sum_{\mu // H} \mathfrak{m}_\mu$$

Proposition

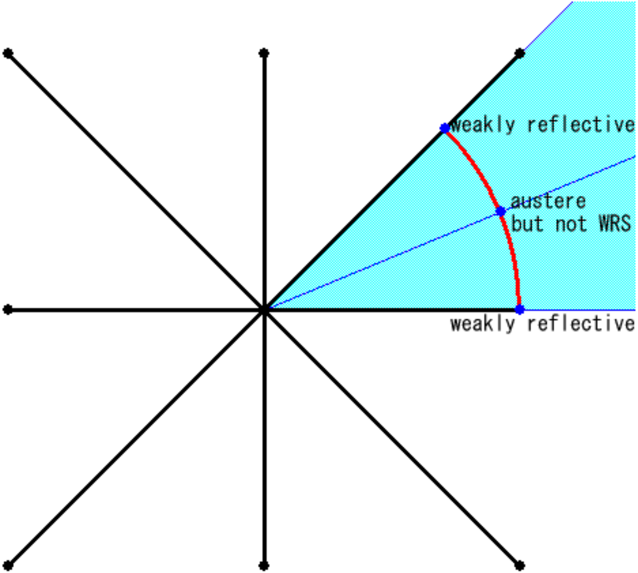
For a resticted root λ, the orbit $\mathrm{Ad}(K)\lambda$ is tangentially degenerate

$$\iff \sum_{\mu // \lambda} \mathfrak{m}_\mu \text{ has non-zero subspace invariant under } \mathrm{ad}(\mathfrak{z}_K^\lambda).$$

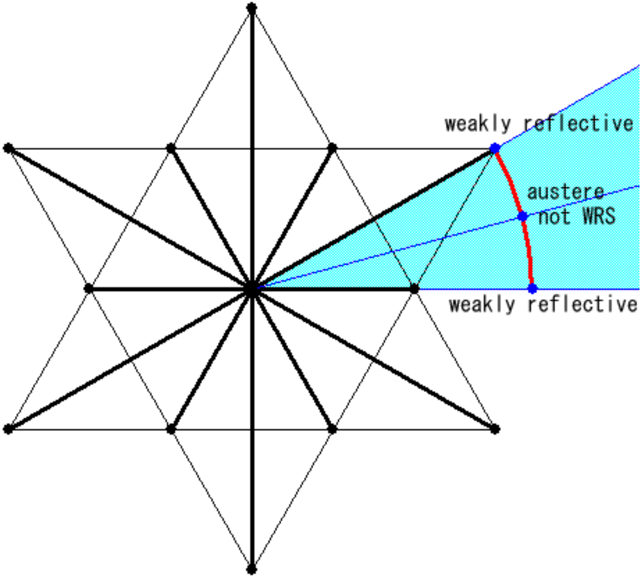
Case of type A_2



Case of type B_2



Case of type G_2



Related problems

- Ishikawa-Kimura-Miyaoka's problem on the Ferus inequality.
- Classify all weakly reflective submanifolds in Riemannian symmetric spaces.
- Homogeneity of weakly reflective submanifolds. Construct examples of non-homogeneous weakly reflective submanifolds.
- Study geometry of special Lagrangian submanifolds obtained from austere submanifolds via twisted normal bundle.

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