Tight Lagrangian submanifolds in some homogeneous Kähler manifolds

(Joint work with Hiroshi Iriyeh)

Takashi Sakai

Tokyo Metropolitan University

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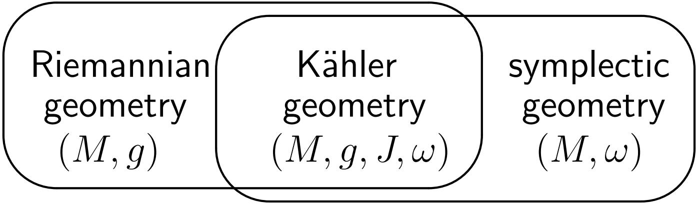
Introduction



$L \subset (M, g, J, \omega)$: **Lagrangian submanifold**

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \dim L = \frac{1}{2} \dim M \\ \omega|_L \equiv 0 \end{cases} \Longleftrightarrow \begin{cases} \dim L = \frac{1}{2} \dim M \\ J(T_pL) \perp T_pL \quad (\forall p \in L) \end{cases}$$

Y.-G. Oh

- Hamiltonian volume minimizing property
- Tightness

Definition of tightness

G/K : homogeneous Kähler manifold

$L \subset G/K$: closed embedded Lagrangian submanifold

Definition

L : **globally tight** (resp. **locally tight**)

$$\overset{\text{def}}{\Longleftrightarrow} \qquad \#(L \cap gL) = SB(L, \mathbb{Z}_2) := \sum_{i=0}^{\dim L} \operatorname{rank} H_i(L; \mathbb{Z}_2)$$

for $\forall g \in G$ (resp. $g \in G$ close to the identity)

where L and gL transversally intersect.

Problem 1

Classify all possible locally tight (or globally tight) Lagrangian submanifolds in Hermitian symmetric spaces.

Lagrangian intersection theory

Definition

M : Hermitian manifold

$L \subset M$: **real form**

$\overset{\text{def}}{\Longleftrightarrow} \exists \tau$: anti-holomorphic involutive isometry of M s.t.

$$L = \{x \in M \mid \tau(x) = x\}$$

Arnold-Givental inequality (Y.-G. Oh)

G/K : Hermitian symmetric space of compact type

$L \subset G/K$: real form

Assume that minimal Maslov number $\Sigma_L \geq 2$

$\Longrightarrow$ $$\#(L \cap \phi L) \geq SB(L, \mathbb{Z}_2)$$

for $\forall \phi \in \mathrm{Ham}(G/K)$ where L and ϕL transversally intersect.

Tight Lagrangian submanifolds

Theorem (Y.-G. Oh, 1991)

$L \subset \mathbb{C}P^n$: *locally tight Lagrangian submanifold*

$\Longrightarrow$ L *must be congruent to*

- *totally geodesic* $\mathbb{R}P^n \subset \mathbb{C}P^n$, *when* $n \geq 2$
- *a latitude circle in* $S^2 \cong \mathbb{C}P^1$, *when* $n = 1$

Theorem (Iriyeh-S., 2010)

$L \subset S^2(1) \times S^2(1) \cong Q_2(\mathbb{C})$: *locally tight Lagrangian submanifold*

$\Longrightarrow$ L *must be congruent to one of*

1. $\mathbf{M}_0 := \{(x, -x) \mid x \in S^2(1)\} \subset S^2(1) \times S^2(1)$
2. $T_{a,b} := S^1(a) \times S^1(b) \subset S^2(1) \times S^2(1) \qquad (0 < a, b \leq 1)$

Strategy to prove Theorem 1

$M \subset (\widetilde{M}, \langle\ ,\ \rangle)$: submanifold

$\mathfrak{i}(\widetilde{M})$: Lie algebra of all Killing vector fields of $\widetilde{M}$

$$\mathfrak{i}(\widetilde{M})^{NM} := \{Z^{NM} \in \Gamma(NM) \mid Z \in \mathfrak{i}(\widetilde{M})\}$$

$\mathrm{nul}_K(M) := \dim \mathfrak{i}(\widetilde{M})^{NM}$: **Killing nullity** of M

For $p \in M$

$$\begin{array}{rcl} \Phi_p : \mathfrak{i}(\widetilde{M})^{NM} & \longrightarrow & N_pM \oplus \mathrm{Hom}(T_pM, N_pM) \\ Z^{NM} & \longmapsto & (Z_p^{NM}, \nabla^{NM} Z^{NM}) \end{array}$$

$$\boxed{\mathrm{nul}_K(M) \geq \dim \mathrm{Im}\Phi_p}$$

Gotoh's inequality

G/K : Riemannian symmetric space

$$\mathfrak{g} = \mathfrak{k} + \widetilde{\mathfrak{m}} \quad \text{(canonical decomposition)}$$

$M \subset G/K$: compact submanifold, suppose $M \ni o := K$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} + \mathfrak{m}^{\perp}$$

$$\begin{aligned} \Psi_1 : \mathfrak{g} &\longrightarrow \mathfrak{m}^{\perp} \\ Z &\longmapsto Z^{\perp} \\ \Psi_2 : \mathfrak{g} &\longrightarrow \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp}) \end{aligned}$$

$$\Psi_2(Z)(X) := (\mathrm{ad}_{\mathfrak{g}}(Z_{\mathfrak{k}})X)^{\perp} - B(X, Z_{\mathfrak{m}}) \qquad (X \in \mathfrak{m})$$

where $B : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}^{\perp}$: second fundamental form of M at o

Gotoh's inequality

$$
\begin{array}{ccccc}
\mathfrak{g} & & \xrightarrow{\quad \Psi := \Psi_1 \oplus \Psi_2 \quad} & & \mathfrak{m}^{\perp} \oplus \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp}) \\
\Big\downarrow \Pi & & & & \Big\downarrow \cong \\
\mathfrak{i}(G/K) & \xrightarrow{P} & \mathfrak{i}(G/K)^{NM} & \xrightarrow{\Phi_o} & N_oM \oplus \mathrm{Hom}(T_oM, N_oM)
\end{array}
$$

Theorem (Gotoh, 1999)

$M \subset G/K$: *compact connected submanifold,* *suppose* $M \ni o$

$$\Longrightarrow \qquad \mathrm{nul}_K(M) \geq \mathrm{codim}(M) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}})$$

Moreover, the equality holds if and only if $M \subset G/K$ *is a homogeneous submanifold.*

An estimate for the case of $S^2 \times S^2$

Proposition

$L \subset S^2 \times S^2$: compact connected Lagrangian surface

Suppose $L \ni o := (e_1, e_1) \in S^2 \times S^2$

$$\Longrightarrow \qquad \mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}}) \geq 3.$$

Moreover, the equality of the second inequality holds if and only if $\mathfrak{m} \subset \widetilde{\mathfrak{m}}$ is a complex subspace w.r.t. $J_0 \oplus (-J_0)$.

$$\mathfrak{i}(\widetilde{M}) \cong \mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

Hence, possible Killing nullities are 3, 4, 5 and 6. Moreover, we can prove that L is homogeneous when $\mathrm{nul}_K(L) = 3$ or 4.

Nonexistence of large Killing nullities

Assume that $\mathrm{nul}_K(L) = 6$

$W_1, \dots, W_6 \in \mathfrak{i}(G/K)$ s.t $\{W_1^{NL}, \dots, W_6^{NL}\}$ basis of $\mathfrak{i}(G/K)^{NL}$

For $i = 1, 2, \dots, 6$,

$$\exists f_i \in C^\infty(G/K) \quad \text{s.t.} \quad df_i = \omega(W_i, \cdot)$$

$$\phi_i := \iota^* f_i \in C^\infty(L)$$

If $L \subset G/K$ is a tight Lagrangian submanifold, then $\phi : L \longrightarrow \mathbb{E}^6$

$$\phi(p) := (\phi_1(p), \phi_2(p), \dots, \phi_6(p)) \quad (p \in L)$$

is a tight map.

Theorem (Kuiper, Little-Pohl)

Let M^n be a closed n-dimensional manifold. If $\phi : M^n \to \mathbb{E}^N$ is a tight smooth map substantially into $\mathbb{E}^N$, then

$$N \leq \frac{1}{2} n(n+3).$$

Global tightness of real forms

Theorem (Takeuchi-Kobayashi, 1968)

Any real form in a Hermitian symmetric space of compact type is locally tight.

Problem 2

Is any real form of a Hermitian symmetric space of compact type globally tight?

Theorem (Howard, 1993)

A totally geodesic Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$ *is globally tight.*

Global tightness of real forms

G/K : Riemannian symmetric space

$S \subset G/K$: **antipodal set** $\overset{\text{def}}{\iff}$ $\forall x \in S, \quad s_x(y) = y \ (\forall y \in S)$

two number $\#_2(G/K)$ = maximal cardinality of antipodal sets

Example

$\{\mathbb{R}e_0, \mathbb{R}e_1, \dots, \mathbb{R}e_n\} \subset \mathbb{R}P^n$ is a great antipodal set.

Theorem (Tasaki, Tanaka-Tasaki)

G/K : *Hermitian symmetric space of compact type*

$L_1, L_2 \subset G/K$: *real forms which transversally intersect*

$\implies L_1 \cap L_2$ *is an antipodal set of* L_1 *and* L_2

In addition, if L_1 *and* L_2 *are congruent to each other*

$\implies L_1 \cap L_2$ *is a great antipodal set of* L_1 *and* L_2

$$\#(L_1 \cap L_2) = \#_2(L_1) = \#_2(L_2)$$

Global tightness of real forms

By M. Takeuchi's results

1. any real form L of a Hermitian symmetric space G/K is a symmetric R-space
2. M : symmetric R-space $\implies$ $\#_2(M) = SB(M;\mathbb{Z}_2)$

Corollary

Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.

Real forms of complex hyperquadrics

$$\begin{aligned} Q_n(\mathbb{C}) &= \{[z] \in \mathbb{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+2}^2 = 0\} \\ Q_{k,n-k}(\mathbb{R}) &= \{[x] \in \mathbb{R}P^{n+1} \mid x_0^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{n+1}^2 = 0\} \end{aligned}$$

$$\begin{aligned} Q_{k,n-k}(\mathbb{R}) &\hookrightarrow Q_n(\mathbb{C}) \\ [x_0, \ldots, x_{n+1}] &\longmapsto [x_0, \ldots, x_k, \sqrt{-1}x_{k+1}, \ldots, \sqrt{-1}x_{n+1}] \end{aligned}$$

$$\begin{aligned} Q_n(\mathbb{C}) &\cong \widetilde{G}_2(\mathbb{R}^{n+2}) \subset \wedge^2\mathbb{R}^{n+2} \\ [x + \sqrt{-1}y] &\longleftrightarrow \{x, y\}_{\mathbb{R}} = x \wedge y \end{aligned}$$

$$\begin{aligned} Q_{k,n-k}(\mathbb{R}) &= S^k(\{e_0, \ldots, e_k\}_{\mathbb{R}}) \wedge S^{n-k}(\{e_{k+1}, \ldots, e_{n+1}\}_{\mathbb{R}}) \\ &\cong (S^k \times S^{n-k})/\mathbb{Z}_2 \end{aligned}$$

Real forms of complex hyperquadrics

Case of $n = 2$

$$Q_2(\mathbb{C}) \cong \widetilde{G}_2(\mathbb{R}^4) \cong S^2 \times S^2$$

$$Q_{0,2}(\mathbb{R}) = S^2 = \mathbf{M}_0, \qquad Q_{1,1}(\mathbb{R}) = (S^1 \times S^1)/\mathbb{Z}_2 = T_{1,1}$$

Theorem (Tasaki)

$0 \leq k \leq l \leq [n/2]$

$L_1, L_2 \subset Q_n(\mathbb{C})$: *real forms congruent to* $Q_{k,n-k}(\mathbb{R}), Q_{l,n-l}(\mathbb{R})$

If L_1 *and* L_2 *transversally intersect,*

$$\Longrightarrow L_1 \cap L_2 \cong \{\pm e_0 \wedge e_{k+1},\ \pm e_1 \wedge e_{k+2},\ \ldots,\ \pm e_k \wedge e_{2k+1}\}$$

$$\#(L_1 \cap L_2) = 2k + 2 = \#_2(L_1) = SB(L_1; \mathbb{Z}_2)$$

Real forms of partial flag manifolds

$$\begin{aligned} F_{k_1,\dots,k_r}(\mathbb{C}^n) &= \left\{ (V_1,\dots,V_r) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_i = k_1 + \cdots + k_i \\ V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n \end{array} \right\} \\ &\cong U(n)/(U(k_1)\times\cdots\times U(k_r)) \\ F_{k_1,\dots,k_r}(\mathbb{R}^n) &\cong O(n)/(O(k_1)\times\cdots\times O(k_r)) \end{aligned}$$

$$\begin{array}{ccc} F_{k_1,\dots,k_r}(\mathbb{R}^n) & \hookrightarrow & F_{k_1,\dots,k_r}(\mathbb{C}^n) \\ (V_1,\dots,V_r) & \longmapsto & (V_1^{\mathbb{C}},\dots,V_r^{\mathbb{C}}) \end{array}$$

$V \subset \mathbb{C}^n$: Lagrangian subspace

$\exists v_1,\cdots,v_n$: o.n.b of $\mathbb{R}^n$, $\exists \theta_1,\cdots,\theta_n \in \mathbb{R}$ s.t.

$$V = \{e^{\sqrt{-1}\theta_1}v_1,\dots,e^{\sqrt{-1}\theta_n}v_n\}_{\mathbb{R}}$$

Any Lagrangian submanifold congruent to $F_{k_1,\dots,k_r}(\mathbb{R}^n)$ can be

$$gF_{k_1,\dots,k_r}(\mathbb{R}^n) = F_{k_1,\dots,k_r}(g\mathbb{R}^n), \quad g \in U(n)$$

Real forms of partial flag manifolds

Theorem (Iriyeh-Tasaki-S.)

$L = F_{k_1,\dots,k_r}(\mathbb{R}^n) \subset F_{k_1,\dots,k_r}(\mathbb{C}^n)$

If L and gL transversally intersect

$$\begin{aligned} \Longrightarrow L \cap gL &= F_{k_1,\dots,k_r}(\mathbb{R}^n) \cap F_{k_1,\dots,k_r}(g\mathbb{R}^n) \\ &= \Big\{ (\{v_{i_1},\dots,v_{i_{k_1}}\}_{\mathbb{C}},\cdots,\{v_{i_1},\dots,v_{i_{k_1+\cdots+k_r}}\}_{\mathbb{C}}) \\ &\qquad \Big| \{i_1,i_2,\dots,i_n\} = \{1,2,\dots,n\} \Big\} \end{aligned}$$

$$\#(L \cap gL) = \frac{n!}{k_1! \cdot k_2! \cdots k_r! \cdot k_{r+1}!} = SB(L;\mathbb{Z}_2)$$

Corollary

A real form $F_{k_1,\dots,k_r}(\mathbb{R}^n) \subset F_{k_1,\dots,k_r}(\mathbb{C}^n)$ is a globally tight Lagrangian submanifold.

Hamiltonian volume minimizing property

(M, ω) : symplectic submanifold

$F : M \times [0,1] \longrightarrow \mathbb{R}$

$\rightsquigarrow X_t \in \mathfrak{X}(M)$: **Hamiltonian vector field**

$$\omega(X_t, \ \cdot\ ) = dF_t(\ \cdot\ )$$

$\rightsquigarrow \phi_t \in \mathrm{Diff}(M)$: **Hamiltonian isotopy**

$$\frac{d\phi_t}{dt} = X_t \circ \phi_t, \quad \phi_0 = \mathrm{id}_M$$

$$\mathrm{Ham}(M, \omega) := \{\text{Hamiltonain isotopy of } M\}$$

Definition

(M, g, ω) : Kähler manifold

$L \subset M$: **Hamiltonian volume minimizing**

$$\overset{\text{def}}{\Longleftrightarrow} \mathrm{vol}(L) \leq \mathrm{vol}(\phi L) \quad \text{for} \quad \forall \phi \in \mathrm{Ham}(M, \omega)$$

Hamiltonian volume minimizing property

Theorem (Kleiner-Oh, 1990)

Let $L = \mathbb{R}P^n \subset \mathbb{C}P^n$

$\Longrightarrow \quad \mathrm{Vol}(L) \leq \mathrm{Vol}(\phi L)$ *for* $\forall \phi \in \mathrm{Ham}(\mathbb{C}P^n)$

Theorem 3 (Iriyeh-H. Ono-S., 2003)

Let $L = S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$

$\Longrightarrow \quad \mathrm{Vol}(L) \leq \mathrm{Vol}(\phi L)$ for $\forall \phi \in \mathrm{Ham}(S^2 \times S^2)$

Problem 3

Does a globally tight Lagrangian submanifold have Hamiltonian volume minimizing property?

Real forms in $Q_n(\mathbb{C})$ (Summary)

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

Real forms in $Q_n(\mathbb{C})$ (Summary)

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

globally tight (Iriyeh-S., Tasaki)

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

Hamiltonian volume minimizing (Iriyeh-Ono-S.)

H-stable (Oh, Amarzaya-Ohnita)

globally tight (Iriyeh-S., Tasaki)

H-unstable (Oh, Amarzaya-Ohnita)

homologically volume minimizing (Gluck-Morgan-Ziller, Lê)

Problem 4

Are there any globally tight Lagrangian submanifolds except real forms?