Lagrangian Floer homology and its application to Hamiltonian volume minimizing property

(Joint work with Hiroshi Iriyeh and Hiroyuki Tasaki)

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Introduction

Arnold-Givental inequality (Y.-G. Oh)

(M, J_0, ω) : (irreducible) Herm. symm. space of compact type

$\sigma : M \longrightarrow M$: anti-holomorphic involution

$$L := \mathrm{Fix}(\sigma) \quad \textbf{real form}$$

$$\Longrightarrow \qquad \#(L \cap \phi L) \geq SB(L, \mathbb{Z}_2) := \sum \mathrm{rank}\, H_i(L, \mathbb{Z}_2)$$

for any $\phi \in \mathrm{Ham}(M, \omega)$ with $L \pitchfork \phi L$.

Lagrangian Floer homology

- $HF(L, \phi L : \mathbb{Z}_2) \cong H_*(L, \mathbb{Z}_2)$
- $HF(L, \phi L : \mathbb{Z}_2)$ is invariant under $\phi \in \mathrm{Ham}(M, \omega)$

Plan of this talk

Problem (Y.-G. Oh)

Study the Lagrangian Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ of a pair of real forms (L_0, L_1) in a Hermitian symmetric space M of compact type in the case where L_0 is not necessarily congruent to L_1.

1. Lagrangian Floer homology
2. Floer homology for a pair of real forms in a Hermitian symmetric space of compact type
3. Generalized Arnold-Givental inequality
4. Volume estimate for a real form under Hamiltonian deformations

Lagrangian Floer homology

(M, ω) : closed symplectic manifold

$J = \{J_t\}_{0 \leq t \leq 1}$: family of ω-compatible almost complex structures

L_0, L_1 : closed Lagrangian submanifolds, $L_0 \pitchfork L_1$

Definition

$p, q \in L_0 \cap L_1$

$u : \mathbb{R} \times [0,1] \longrightarrow M$: J-**holomorphic strip** from p to q

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \bar{\partial}_J u := \frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = 0 \\ u(s,0) \in L_0, \quad u(s,1) \in L_1 \\ u(-\infty, t) = p, \quad u(+\infty, t) = q \end{cases}$$

Lagrangian Floer homology

$$CF(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 \, p$$

$\partial : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) \cdot q$$

$n(p, q) := \#\{$isolated J-holomorphic strips from p to $q\} \pmod 2$

$$\partial \circ \partial = 0 \quad \Longrightarrow \quad HF(L_0, L_1 : \mathbb{Z}_2) := \ker \partial / \mathrm{im} \partial$$

1. $HF(L_0, L_1 : \mathbb{Z}_2)$ is independent of J
2. $HF(\phi L_0, \psi L_1 : \mathbb{Z}_2) \cong HF(L_0, L_1 : \mathbb{Z}_2)$
   for $\forall \phi, \psi \in \mathrm{Ham}(M, \omega)$
3. $HF(L, \phi L : \mathbb{Z}_2) \cong H_*(L, \mathbb{Z}_2)$

Lagrangian Floer homology

Theorem (Oh)

L_0, L_1 : monotone, minimal Maslov number $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

$\Longrightarrow$

1. $HF(L_0, L_1 : \mathbb{Z}_2)$ *is well-defined*
2. $HF(L_0, L_1 : \mathbb{Z}_2) \cong HF(L_0, \phi L_1 : \mathbb{Z}_2)$ *for* $\forall \phi \in \mathrm{Ham}(M, \omega)$

Hence if $L_0 \pitchfork L_1$*,*

$$\#(L_0 \cap L_1) \geq \operatorname{rank} HF(L_0, L_1 : \mathbb{Z}_2)$$

Floer homology for a pair of real forms

Theorem 1 (Iriyeh-Tasaki-S.)

(M, J_0, ω) : monotone Hermitian symmetric space of compact type

L_0, L_1 : real forms, $L_0 \pitchfork L_1$, $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

$\Longrightarrow$

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p]$$

1. (M, J_0, ω) is monotone if and only if it is Kähler-Einstein.
2. If M is irreducible, then the assumptions are satisfied except for the case $\mathbb{R}P^1 \subset \mathbb{C}P^1$.

Antipodal set

M : Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$

$S \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y \ (\forall x, y \in S)$

$\#_2 M := \sup\{\#S \mid S$: antipodal set of $M\}$: **2-number** of M

Theorem (Takeuchi)

M : *symmetric R-space* $\Longrightarrow$ $\#_2 M = SB(M, \mathbb{Z}_2)$

Theorem (Tanaka-Tasaki)

M : *Hermitian symmetric space of compact type*

L_0, L_1 : *real forms of M, $L_0 \pitchfork L_1$*

$\Longrightarrow$ $L_0 \cap L_1$ *is an antipodal set of L_0 and L_1.*

Real forms of irreducible Hermitian symmetric spaces

M	L_0	L_1
$G^{\mathbb{C}}_{2q}(\mathbb{C}^{2m+2q})$	$G^{\mathbb{H}}_{q}(\mathbb{H}^{m+q})$	$G^{\mathbb{R}}_{2q}(\mathbb{R}^{2m+2q})$
$G^{\mathbb{C}}_{n}(\mathbb{C}^{2n})$	$U(n)$	$G^{\mathbb{R}}_{n}(\mathbb{R}^{2n})$
$G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m})$	$G^{\mathbb{H}}_{m}(\mathbb{H}^{2m})$	$U(2m)$
$SO(4m)/U(2m)$	$U(2m)/Sp(m)$	$SO(2m)$
$Sp(2m)/U(2m)$	$Sp(m)$	$U(2m)/O(2m)$
$Q_n(\mathbb{C})$	$S^{k,n-k}$	$S^{l,n-l}$
$E_6/T \cdot Spin(10)$	$F_4/Spin(9)$	$G^{\mathbb{H}}_{2}(\mathbb{H}^4)/\mathbb{Z}_2$
$E_7/T \cdot E_6$	$T \cdot (E_6/F_4)$	$(SU(8)/Sp(4))/\mathbb{Z}_2$

$$S^{k,n-k} = (S^k \times S^{n-k})/\mathbb{Z}_2$$

Cases of irreducible Hermitian symmetric spaces

Theorem 2 (Iriyeh-Tasaki-S.)

M : irreducible Hermitian symmetric space of compact type

L_0, L_1 : real forms of M, $L_0 \pitchfork L_1$

$\Longrightarrow$

1. $(M, L_0, L_1) \cong (G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m}),\ G^{\mathbb{H}}_m(\mathbb{H}^{2m}),\ U(2m)) \quad (m \geq 2)$

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2^m}$$

   where $2^m < \binom{2m}{m} = \#_2 L_0 < 2^{2m} = \#_2 L_1$

2. (M, L_0, L_1) : otherwise

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{\min\{\#_2 L_0, \#_2 L_1\}}$$

Generalized Arnold-Givental inequality

Corollary 3

M : irreducible Hermitian symmetric space of compact type

(L_0, L_1) : real forms of M

$\Longrightarrow$ for any $\phi \in \mathrm{Ham}(M, \omega)$, $L_0 \pitchfork \phi L_1$

1. $(M, L_0, L_1) \cong (G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m}),\ G^{\mathbb{H}}_m(\mathbb{H}^{2m}),\ U(2m)) \quad (m \geq 2)$

$$\#(L_0 \cap \phi L_1) \geq 2^m$$

2. (M, L_0, L_1) : otherwise

$$\#(L_0 \cap \phi L_1) \geq \min\{SB(L_0, \mathbb{Z}_2), SB(L_1, \mathbb{Z}_2)\}$$

Hamiltonian volume minimizing property

(M, ω) : compact symplectic submanifold

$F : M \times [0,1] \longrightarrow \mathbb{R}$

$\rightsquigarrow X_t \in \mathfrak{X}(M)$: **Hamiltonian vector field**

$$\omega(X_t, \cdot\,) = dF_t(\,\cdot\,)$$

$\rightsquigarrow \phi_t \in \mathrm{Diff}(M)$: **Hamiltonian isotopy**

$$\frac{d\phi_t}{dt} = X_t \circ \phi_t, \quad \phi_0 = \mathrm{id}_M$$

$$\mathrm{Ham}(M, \omega) := \{\phi_1 \mid \phi_t : \text{Hamiltonian isotopy of } M\}$$

Definition

(M, g, ω) : Kähler manifold

$L \subset M$: **Hamiltonian volume minimizing**

$$\overset{\text{def}}{\Longleftrightarrow} \quad \mathrm{vol}(L) \leq \mathrm{vol}(\phi L) \quad \text{for} \quad \forall \phi \in \mathrm{Ham}(M, \omega)$$

Hamiltonian volume minimizing property

Example

- $\mathbb{R}P^n \subset \mathbb{C}P^n$ (Kleiner-Oh, 1990)
- $S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$ (Iriyeh-Ono-S., 2003)

Real forms of complex hyperquadrics

$$\begin{aligned} Q_n(\mathbb{C}) &= \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + z_2^2 + \cdots + z_{n+2}^2 = 0\} \\ S^{k,n-k} &= \{[x] \in \mathbb{R}P^{n+1} \mid x_1^2 + \cdots + x_{k+1}^2 - x_{k+2}^2 - \cdots - x_{n+2}^2 = 0\} \end{aligned}$$

$$\begin{aligned} S^{k,n-k} &\hookrightarrow Q_n(\mathbb{C}) \\ [x_1, \ldots, x_{n+2}] &\longmapsto [x_1, \ldots, x_{k+1}, \sqrt{-1}x_{k+2}, \ldots, \sqrt{-1}x_{n+2}] \end{aligned}$$

$$\begin{aligned} Q_n(\mathbb{C}) &\cong \widetilde{G}_2(\mathbb{R}^{n+2}) \subset \wedge^2\mathbb{R}^{n+2} \\ [x + \sqrt{-1}y] &\longleftrightarrow \{x, y\}_{\mathbb{R}} = x \wedge y \end{aligned}$$

$$\begin{aligned} S^{k,n-k} &= S^k(\{e_1, \ldots, e_{k+1}\}_{\mathbb{R}}) \wedge S^{n-k}(\{e_{k+2}, \ldots, e_{n+2}\}_{\mathbb{R}}) \\ &\cong (S^k \times S^{n-k})/\mathbb{Z}_2 \end{aligned}$$

Volume estimate under Hamiltonian deformations

Theorem 4 (Iriyeh-Tasaki-S.)

$\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$ for $\forall \phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$

Corollary 5

$S^{0,n} \subset Q_n(\mathbb{C})$ is Hamiltonian volume minimizing.

Theorem (Gluck-Morgan-Ziller)

$S^{0,n} \subset Q_n(\mathbb{C}) \cong \widetilde{G}_n(\mathbb{R}^{n+2})$ is volume minimizing in its homology class when n is even.

- When n is odd, $S^{0,n} \subset Q_n(\mathbb{C})$ can not be homologically volume minimizing.

Proof of Theorem 4

Theorem (Le)

$N \subset Q_n(\mathbb{C}) \cong \widetilde{G}_n(\mathbb{R}^{n+2})$: n-*dim. submanifold*

$$\Longrightarrow \int_{SO(n+2)} \#(gS^n \cap N) d\mu(g) \leq 2\frac{\mathrm{vol}(SO(n+2))}{\mathrm{vol}(S^n)}\mathrm{vol}(N)$$

Put $N = \phi S^{k,n-k}$ $(k = 0, 1, \ldots, [n/2])$. Then

$$\begin{aligned}
\mathrm{vol}(\phi S^{k,n-k}) &\overset{\text{Le}}{\geq} \frac{\mathrm{vol}(S^n)}{2\mathrm{vol}(SO(n+2))} \int_{SO(n+2)} \#(gS^n \cap \phi S^{k,n-k}) d\mu(g) \\
&\overset{\text{GAG}}{\geq} \frac{\mathrm{vol}(S^n)}{2\mathrm{vol}(SO(n+2))} \int_{SO(n+2)} 2 d\mu(g) \\
&= \mathrm{vol}(S^n).
\end{aligned}$$

Real forms in $Q_n(\mathbb{C})$

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

Hamiltonian volume minimizing (Iriyeh-Ono-S.)

H-stable (Oh, Amarzaya-Ohnita)

Ham. vol. min. (Iriyeh-Tasaki-S.)

H-unstable (Oh, A-O)

homologically volume minimizing (Gluck-Morgan-Ziller, Lê)