Tight Lagrangian surfaces in $S^2 \times S^2$

(Joint work with Hiroshi Iriyeh)

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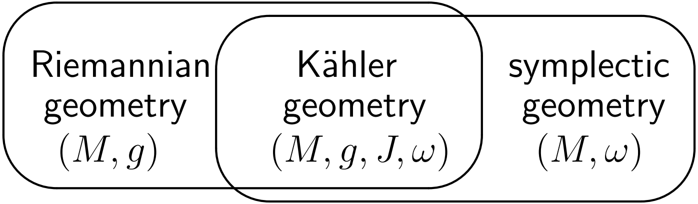
Introduction



$L \subset (M, g, J, \omega)$: Lagrangian submanifold

$$\overset{\text{def}}{\iff} \begin{cases} \dim L = \frac{1}{2} \dim M \\ \omega|_L \equiv 0 \end{cases} \iff \begin{cases} \dim L = \frac{1}{2} \dim M \\ J(T_pL) \perp T_pL \quad (\forall p \in L) \end{cases}$$

Y.-G. Oh

- Hamiltonial volume minimizing property
- Tightness

Definition of tightness

G/K : compact Hermitian symmetric space

$L \subset G/K$: closed embedded Lagrangian submanifold

Definition

L : **globally tight** (resp. **locally tight**)

$$\overset{\text{def}}{\Longleftrightarrow} \quad \#(L \cap gL) = SB(L, \mathbb{Z}_2) := \sum_{i=0}^{\dim L} \operatorname{rank} H_i(L; \mathbb{Z}_2)$$

for $\forall g \in G$ (resp. $g \in G$ close to the identity)

s.t. L transversely intersects with gL.

Problem 1

Classify all possible locally tight (or globally tight) Lagrangian submanifolds in compact Hermitian symmetric spaces.

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Tight Lagrangian submanifolds

Theorem (Y.-G. Oh, 1991)

$L \subset \mathbb{C}P^n$: *locally tight Lagrangian submanifold*

$\Longrightarrow$ *L must be congruent to*

- *totally geodesic* $\mathbb{R}P^n \subset \mathbb{C}P^n$, *when* $n \geq 2$
- *a latitude circle in* $S^2 \cong \mathbb{C}P^1$, *when* $n = 1$

Theorem 1 (Iriyeh-S., 2008)

$L \subset S^2(1) \times S^2(1) \cong Q_2(\mathbb{C})$: locally tight Lagrangian submanifold

$\Longrightarrow$ L must be congruent to one of

1. $\mathrm{M}_0 := \{(x, -x) \mid x \in S^2(1)\} \subset S^2(1) \times S^2(1)$
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Strategy to prove Theorem 1

$M \subset (\widetilde{M}, \langle\ ,\ \rangle)$: submanifold
$\mathfrak{i}(\widetilde{M})$: Lie algebra of all Killing vector fields of $\widetilde{M}$

$$\mathfrak{i}(\widetilde{M})^{NM} := \{Z^{NM} \in \Gamma(NM) \mid Z \in \mathfrak{i}(\widetilde{M})\}$$

$\mathrm{nul}_K(M) := \dim \mathfrak{i}(\widetilde{M})^{NM}$: **Killing nullity** of M

For $p \in M$

$$\begin{aligned} \Phi_p : \mathfrak{i}(\widetilde{M})^{NM} &\longrightarrow N_pM \oplus \mathrm{Hom}(T_pM, N_pM) \\ Z^{NM} &\longmapsto (Z_p^{NM}, \nabla^{NM} Z^{NM}) \end{aligned}$$

$$\boxed{\mathrm{nul}_K(M) \geq \dim \mathrm{Im}\Phi_p}$$

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Gotoh's inequality

G/K : Riemannian symmetric space

$$\mathfrak{g} = \mathfrak{k} + \widetilde{\mathfrak{m}} \quad \text{(canonical decomposition)}$$

$M \subset G/K$: compact submanifold, suppose $M \ni o := K$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} + \mathfrak{m}^{\perp}$$

$$\begin{aligned} \Psi_1 : \mathfrak{g} &\longrightarrow \mathfrak{m}^{\perp} \\ Z &\longmapsto Z^{\perp} \\ \Psi_2 : \mathfrak{g} &\longrightarrow \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^{\perp}) \end{aligned}$$

$$\Psi_2(Z)(X) := (\mathrm{ad}_{\mathfrak{g}}(Z_{\mathfrak{k}})X)^{\perp} - B(X, Z_{\mathfrak{m}}) \qquad (X \in \mathfrak{m})$$

where $B : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}^{\perp}$: second fundamental form of M at o

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Gotoh's inequality

$$\begin{array}{ccccc} \mathfrak{g} & & \xrightarrow{\quad \Psi := \Psi_1 \oplus \Psi_2 \quad} & & \mathfrak{m}^\perp \oplus \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^\perp) \\ \Pi \downarrow & & & & \downarrow \cong \\ \mathfrak{i}(G/K) & \xrightarrow{P} & \mathfrak{i}(G/K)^{NM} & \xrightarrow{\Phi_o} & N_oM \oplus \mathrm{Hom}(T_oM, N_oM) \end{array}$$

Theorem (Gotoh, 1999)

$M \subset G/K$: *compact connected submanifold,* *suppose* $M \ni o$

$$\Longrightarrow \qquad \mathrm{nul}_K(M) \geq \mathrm{codim}(M) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}})$$

Moreover, the equatily holds if and only if $M \subset G/K$ *is a homogeneous submanifold.*

An estimate for the case of $S^2 \times S^2$

Proposition

$L \subset S^2 \times S^2$: compact connected Lagrangian surface

Suppose $L \ni o := (e_1, e_1) \in S^2 \times S^2$

$$\Longrightarrow \qquad \mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}}) \geq 3.$$

Moreover, the equality of the second inequality holds if and only if $\mathfrak{m} \subset \widetilde{\mathfrak{m}}$ is a complex subspace w.r.t. $J_0 \oplus (-J_0)$.

$$\mathfrak{i}(\widetilde{M}) \cong \mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

Hence, possible Killing nullities are 3, 4, 5 and 6.

The case where $\mathrm{nul}_K(L) = 3$

$$3 = \mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}}) \geq 3$$

for all points of L. Thus L must be congruent to $\mathbf{M}_0$.

The case where $\mathrm{nul}_K(L) = 4$

Since $L \not\cong \mathbf{M}_0$, $\exists p \in L, \exists g \in G$ s.t. $gp = o$ and

$$4 = \mathrm{nul}_K(L) = \mathrm{nul}_K(gL) \geq \mathrm{codim}(gL) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{k}}) \geq 4.$$

Theorem (Ma-Ohnita)

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Global tightness of real forms

Theorem (Takeuchi-Kobayashi, 1968)

A real form $L \subset G/K$ is locally tight.

Problem 2

Is a real form $L \subset G/K$ globally tight?

Theorem (Howard, 1993)

A totally geodesic Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$ is globally tight.

Theorem 2 (Iriyeh-S., 2009)

- $\mathrm{M}_0 \subset S^2 \times S^2$ is globally tight.
- A real form $Q_{2,3}(\mathbb{R}) \subset Q_3(\mathbb{C})$ is globally tight.
- Real forms $Q_{1,n+1}(\mathbb{R}) \subset Q_n(\mathbb{C})$ are globally tight.

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Real forms in complex hyperqudrics

$$Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + \cdots + z_{n+2}^2 = 0\}$$

$$\begin{array}{ccc} Q_n(\mathbb{C}) & \cong & \widetilde{G}_2(\mathbb{R}^{n+2}) \\ [x + \sqrt{-1}y] & \longleftrightarrow & x \wedge y \end{array}$$

For $\quad p + q = n + 2$

$$\begin{aligned} Q_{p,q}(\mathbb{R}) &= \{[x] \in \mathbb{R}P^{n+1} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0\} \\ &= S^{p-1} \times S^{q-1}/\mathbb{Z}_2 \\ &\subset Q_n(\mathbb{C}) \end{aligned}$$

Example (Case of $n = 2$)

$$Q_2(\mathbb{C}) \cong \widetilde{G}_2(\mathbb{R}^4) \cong S^2 \times S^2$$

$$Q_{1,3}(\mathbb{R}) = S^2 = \mathbf{M}_0, \qquad Q_{2,2}(\mathbb{R}) = S^1 \times S^1/\mathbb{Z}_2 = T_{1,1}$$

Lagrangian intersection theory and integral geometry

Arnold-Givental inequality (Y.-G. Oh)

G/K : compact Hermitian symmetric space

$L \subset G/K$: real form

Assume that minimal Maslov number $\Sigma_L \geq 2$

$\Longrightarrow \qquad \#(L \cap \phi L) \geq SB(L, \mathbb{Z}_2)$

for $\forall \phi \in \mathrm{Ham}(G/K)$ s.t. L transversely intersects with ϕL.

Generalized Poincaré formula (Howard, 1993)

G/K : Riem. homog. space, assume G is unimodular.

$M, N \subset G/K$: submanifolds, $\dim(G/K) \leq \dim M + \dim N$

$$\Longrightarrow \int_G \mathrm{Vol}(M \cap gN) d\mu(g) = \int_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu(x, y)$$

Outline of the proof of Theorem 2

Assume that $\mathbf{M}_0 \subset S^2 \times S^2$ is not globally tight.

Then, $\exists$ an open nbd $U \subset G$ which satisfies

$$\#(\mathbf{M}_0 \cap g\mathbf{M}_0) \geq SB(\mathbf{M}_0, \mathbb{Z}_2) + 1 \qquad (\forall g \in U)$$

$$\begin{aligned}
2\ \mathrm{vol}(G) &\overset{\text{(P)}}{=} \int_G \#(\mathbf{M}_0 \cap g\mathbf{M}_0) d\mu(g) \\
&= \int_{G \setminus U} \#(\mathbf{M}_0 \cap g\mathbf{M}_0) d\mu(g) + \int_U \#(\mathbf{M}_0 \cap g\mathbf{M}_0) d\mu(g) \\
&\overset{\text{(AG)}}{\geq} \int_{G \setminus U} SB(\mathbf{M}_0, \mathbb{Z}_2) d\mu(g) + \int_U (SB(\mathbf{M}_0, \mathbb{Z}_2) + 1) d\mu(g) \\
&= \int_G SB(\mathbf{M}_0, \mathbb{Z}_2) d\mu(g) + \int_U d\mu(g) \\
&> SB(\mathbf{M}_0, \mathbb{Z}_2)\mathrm{vol}(G) \\
&= 2\ \mathrm{vol}(G)
\end{aligned}$$

Hamiltonian volume minimizing property

Theorem (Kleiner-Oh, 1990)

Let $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

$\Longrightarrow \quad \mathrm{Vol}(L) \leq \mathrm{Vol}(\phi L)$ *for* $\forall \phi \in \mathrm{Ham}(\mathbb{C}P^n)$.

Theorem 3 (Iriyeh-H. Ono-S., 2003)

Let $L = S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$.

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Does a globally tight Lagrangian submanifold have Hamiltonian volume minimizing property?

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Real forms in $Q_n(\mathbb{C})$ (Summary)

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
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$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

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globally tight (Iriyeh-S.)

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Hamiltonian volume minimizing (Iriyeh-Ono-S.)

H-stable (Oh, Amarzaya-Ohnita)

globally tight (Iriyeh-S.)

H-unstable (Oh, A-O)

homologically volume minimizing (Gluck-Morgan-Ziller, Lê)