Explicit expressions of kinematic formulae in Riemannian homogeneous spaces

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Kinematic formula

G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

kinematic formula

$$\int_G I(M \cap gN) d\mu_G(g) = \left\{ \begin{array}{r} \text{geometric invariants} \\ \text{of } M \text{ and } N \end{array} \right\}$$

Kinematic formula

Poincaré formula

M^p, N^q : submanifolds of a real space form G/K

$$\int_G \mathrm{vol}(M \cap gN) d\mu(g) = \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)} \mathrm{vol}(M)\mathrm{vol}(N).$$

Chern-Federer formula

M^p, N^q : submanifolds of a real space form G/K

For $0 \leq 2l \leq p+q-n$,

$$\int_G \mu_{2l}(M \cap gN) d\mu(g) = \sum_{0 \leq k \leq l} C(n,p,q,k,l)\mu_{2k}(M)\mu_{2(l-k)}(N).$$

Integral invariants

G/K : Riemannian homogeneous space

$V \subset T_o(G/K)$: subspace

$M \subset G/K$: submanifold of type V

$\overset{\text{def}}{\iff} \ {}^\exists g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V$ for each $x \in M$

$$\begin{aligned} II(V) &= \left\{ h : V \times V \to V^\perp ; \text{symmetric bilinear} \right\} \\ K(V) &= \{ k \in K \,|\, k_* V = V \} \end{aligned}$$

$K(V)$ acts on $II(V)$ by

$$(kh)(u,v) = k_* h(k_*^{-1} u, k_*^{-1} v) \quad (u, v \in V)$$

$\mathcal{P}$: polynomial on $II(V)$ invariant under $K(V)$

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M$$

Kinematic formula in Riem. homog. spaces (Howard)

G/K : Riemannian homogeneous space

Assume that G is unimodular

$V, W \subset T_o(G/K), \quad \dim V + \dim W \geq \dim G/K$

$\mathcal{P}$: K-invariant homogeneous polynomial on $EII(T_o(G/K))$

Then there exist finite pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ s.t.

1. $\mathcal{Q}_\alpha$: $K(V)$-invariant homogeneous polynomial on $II(V)$
2. $\mathcal{R}_\alpha$: $K(W)$-invariant homogeneous polynomial on $II(W)$
3. $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = \deg \mathcal{P}$ for each α
4. for any submanifolds M of type V and N of type W in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

holds.

Poincaré formula of complex submanifolds

Theorem (Santaló)

$M^p, N^q \subset \mathbb{C}P^n$: complex submanifolds

$$\int_{U(n+1)} \mathit{vol}(M \cap gN)\, d\mu(g) = \frac{\mathit{vol}(U(n+1))\,\mathit{vol}(\mathbb{C}P^{p+q-n})}{\mathit{vol}(\mathbb{C}P^p)\,\mathit{vol}(\mathbb{C}P^q)} \mathit{vol}(M)\,\mathit{vol}(N)$$

Theorem (Kang-Takahashi-Tasaki-S., S.)

G/K : irreducible Hermitian symmetric space, $\dim_{\mathbb{C}}(G/K) = n$

Assume that K acts irreducibly on $\wedge^p T_o(G/K)^{1,0}$.

Then, for any complex submanifolds M and N in G/K with $\dim_{\mathbb{C}}(M) = n - p$, $\dim_{\mathbb{C}}(N) = n - q$ with $p + q \leq n$,

$$\int_G \mathit{vol}(M \cap gN)\, d\mu_G(g) = \frac{(n-p)!(n-q)!\,\mathit{vol}(K)}{n!(n-p-q)!} \mathit{vol}(M)\,\mathit{vol}(N)$$

holds.

The case of real space forms

The space of homogeneous polynomials of degree two on $II(V)$ invariant under $O(V) \times O(V^{\perp})$ is spanned by

$$\mathcal{Q}_1(h) = \sum_{\substack{1 \le i,j \le p \\ p+1 \le k \le n}} \left(h_{ij}^k\right)^2 = \|h\|^2$$

$$\mathcal{Q}_2(h) = \sum_{p+1 \le k \le n} \left(\sum_{1 \le i \le p} h_{ii}^k \right)^2 = p^2 H^2$$

$$\mathcal{W}_2(h) := 2(\mathcal{Q}_2(h) - \mathcal{Q}_1(h))$$

$$\mathcal{U}_p(h) := p\mathcal{Q}_1(h) - \mathcal{Q}_2(h)$$

Theorem (Chern, Federer, Nijenhuis, Howard, Kang-Suh-S.)

Assume that $2 \leq p+q-n$. *Then, for any submanifolds* M^p *and* N^q *of a real space form* G/K,

$$\int_G I^{\mathcal{W}_2}(M \cap gN)d\mu_G(g) = a(p,q,n)I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + a(q,p,n)\mathrm{vol}(M)I^{\mathcal{W}_2}(N)$$

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN)d\mu_G(g) = b(p,q,n)I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + b(q,p,n)\mathrm{vol}(M)I^{\mathcal{U}_q}(N)$$

hold. Here

$$a(p,q,n) = \frac{p+q-n-1}{p-1} \times \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)},$$

$$b(p,q,n) = \frac{(p+q-n+2)(p+q-n-1)}{(p+2)(p-1)} \times \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}.$$

The case where the intersection is a curve

M^p, N^q : submanifolds of a real space form G/K, $p+q=n+1$

$$I^{\kappa^2}(M \cap gN) = \int_{M\cap gN} \kappa^2 d\sigma.$$

Theorem (C. S. Chen, Kang-Suh-S.)

$$\int_G I^{\kappa^2}(M\cap gN)d\mu_G(g)$$
$$= \left(c(p,n)I^{\mathcal{W}_2}(M) + d(p,n)I^{\mathcal{U}_p}(M)\right)\mathrm{vol}(N)$$
$$+\mathrm{vol}(M)\left(c(q,n)I^{\mathcal{W}_2}(N) + d(q,n)I^{\mathcal{U}_q}(N)\right)$$

holds. Here

$$c(p,n) = \frac{2\pi}{p-1}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)},$$
$$d(p,n) = \frac{6\pi}{(p+2)(p-1)}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}.$$

Transfer principle

$G/K, G'/K'$: Riem. homog. spaces, $\dim G = \dim G'$

ρ : $K \to K'$; isomorphism

ψ : $T_o(G/K) \to T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

$$\Longrightarrow \left\{ \begin{array}{r} K(V)\text{-inv. poly.} \\ \text{on } II(V) \end{array} \right\} \overset{\psi}{\cong} \left\{ \begin{array}{r} K'(\psi V)\text{-inv. poly.} \\ \text{on } II(\psi V) \end{array} \right\}$$

$\Longrightarrow$ If for any M of type V and N of type W in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

holds, then the same kinematic formula holds

for M' of type ψV and N' of type ψW in G'/K'.

Transferred kinematic formulae in TPHS

Theorem (S.)

G/K : *two point homogeneous space,*

(namely a Euclidean space or a rank one symmetric space)

M, N : *real hypersurfaces in* G/K

Then the following kinematic formulae hold:

$$\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n) \big(I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{W}_2}(N)\big)$$

$$\int_G I^{\mathcal{U}_{n-2}}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n) \big(I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{U}_{n-1}}(N)\big)$$