INTEGRAL GEOMETRY AND HAMILTONIAN VOLUME MINIMIZING PROPERTY

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1. Introduction

This paper is a survey of our recent works [9], [10], [11] and [20] on integral geometry and its application for some geometric variational problems.

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also right invariant under elements of K, then G/K is a homogeneous space with an invariant Riemannian metric. Consider now two compact submanifolds M and N of G/K, one fixed and the other moving under the action of $g \in G$. Then it becomes a measurable function on G if we assign an *integral invariant* $I(M \cap gN)$ of submanifold $M \cap gN$ for each $g \in G$. It is called the *kinematic formula* that represents the equality between the integral

$$\int_G I(M \cap gN) d\mu_G(g) \tag{1.1}$$

and some geometric invariants of M and N, where $d\mu_G$ is the invariant measure of G. This integral has been studied by many geometers from various viewpoints. In particular, it is called Poincaré formula if we take $\mathrm{vol}(M \cap gN)$ as an integral invariant in (1.1). For example, in the case where M and N are submanifolds of a real space form, this integral is equal to a constant times the product of the volumes of M and N. This was studied by Poincaré, Blaschke, Santaló and others (see [22] for reference). Santaló [21] showed that if M and N are complex submanifolds of a complex projective space, then the Poincaré formula is expressed as a constant times the product of the volumes of two submanifolds.

Furthermore, Chern [4] and Federer [5] obtained a remarkable result stated as follows: Let $G(\mathbb{R}^n)$ be the group of orientation preserving isometries of Euclidian n-space $\mathbb{R}^n$. Assume that $0 \leq 2l \leq p + q - n$. Then for any compact submanifolds M and N of dimensions p and q in $\mathbb{R}^n$, we have the equality

$$\int_{G(\mathbb{R}^n)} \mu_{2l}(M \cap gN) d\mu_{G(\mathbb{R}^n)}(g) = \sum_{k=0}^{l} a(p, q, n, k, l) \mu_{2k}(M) \mu_{2(l-k)}(N) \tag{1.2}$$

where each constant $a(p, q, n, k, l)$ depends only on the indicated parameters. Here μ_{2i} is the integral invariant which appears in coefficients of the Weyl tube formula and the generalized Gauss-Bonnet formula (see [6]). We shall explain this invariant

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in Section 3. C. S. Chen [3] showed that for any closed surfaces M and N in $\mathbb{R}^3$

$$\begin{aligned}(1.3)\quad & \int_{G(\mathbb{R}^3)} \left(\int_{M \cap gN} \kappa^2 ds \right) d\mu_{G(\mathbb{R}^3)}(g) \\ = \quad & \pi^3 \mathrm{vol}(M) \int_N (\|h^N\|^2 + 2(H^N)^2) d\mu_N + \pi^3 \mathrm{vol}(N) \int_M (\|h^M\|^2 + 2(H^M)^2) d\mu_M\end{aligned}$$

where κ and ds are the curvature and the arc-element of the intersection curve $M \cap gN$. We denote by h^X, H^X and $d\mu_X$ the second fundamental form, the mean curvature and the invariant volume element of a submanifold X, respectively. When G is the unitary group $U(n+1)$ acting on the complex projective space $\mathbb{C}P^n$, M and N are complex submanifolds letting $I(M \cap gN)$ be the integral of Chern class leads to the kinematic formula of Shifrin [23].

Afterward Howard [8] defined integral invariants induced from an invariant homogeneous polynomial of the second fundamental form of $M \cap gN$, and he achieved more general kinematic formula, where G is unimodular and acts transitively on the sets of tangent spaces to each of M and N. He put the kinematic formulas listed above into a uniform shape. However, we must have a nontrivial calculation in order to get concrete forms. For a polynomial of degree 0, his formulation gives exactly the Poincaré formula. In Section 1, using his formulation we attempt to give some sufficient conditions that the Poincaré formulas for complex submanifolds in Hermitian homogeneous spaces can be expressed as a constant times of volumes of two submanifolds. On the other hand, we can not consider invariant homogeneous polynomials of odd degree in the case of real space forms. So our interest goes to the concrete form of the kinematic formuls for integral invariants defined by invariant homogeneous polynomial of degree 2. In Section 2, we give explicit expressions of the kinematic formulas of this case, completely.

The kinematic formulas are interesting in their own forms, and furthermore they have been applied to several geometric problems (see [12] and [22] for references). For example, Zhou [26] was obtained an extention of Chen's formula, and applied it to describe a sufficient condition for Hadwiger's containment problem in higher dimensions. The Poincaré formulas have been applied to show some volume minimizing properties of certain submanifolds (see [13], [14] and [24]). In the present paper, we attempt to apply the Poincaré formula to show Hamiltonian volume minimizing properties of minimal Lagrangian submanifolds in Kähler manifolds.

The equator on S^2 has the least length among all its images under area bisecting deformations. This is a well-known theorem by Poincaré and a special case of isoperimetric inequality for closed curves on S^2. This theorem stands in the intersection of symplectic geometry and Riemannian geometry. If we interpret S^2 as a Kähler manifold and the equator as a minimal Lagrangian submanifold, then area bisecting deformations of the equator are nothing but Hamiltonian deformations. Considering S^2 as $\mathbb{C}P^1$ and the equator as a real form $\mathbb{R}P^1 \subset \mathbb{C}P^1$, it is natural to generalize Poincaré's theorem to the case $\mathbb{R}P^n \subset \mathbb{C}P^n$. In 1990, Y.-G. Oh [16] and B. Kleiner actually showed that the standard real form $\mathbb{R}P^n \subset \mathbb{C}P^n$ is Hamiltonian volume minimizing, applying Poincaré formula and Arnold-Givental inequality. In Section 4, we explain an unified method of proving the Hamiltonian volume minimizing properties of real forms in Hermitian symmetric spaces of compact type and pose a conjecture in terms of integral geometry. Finally we show the conjecture in the case $S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$ and establish its Hamiltonian volume minimizing property and the uniqueness modulo isometric group actions.

2. Poincaré formula of complex submanifolds

In this section, we shall explain about the generalized Poincaré formula in Riemannian homogeneous spaces. And we make assertions for Poincaré formula for complex submanifolds in Hermitian manifolds.

We begin with a definition of the angle between subspaces. Let E be a finite dimensional real vector space with an inner product $\langle\cdot,\cdot\rangle$. For two vector subspaces V and W of dimensions p and q in E with $p+q \leq n$, we take orthonormal bases $v_1,\dots,v_p$ and $w_1,\dots,w_q$ of V and W respectively. Then we define $\sigma(V,W)$, the angele between V and W, by

$$\sigma(V,W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|,$$

where

$$\|x_1 \wedge \cdots \wedge x_k\| = \det(\langle x_i, x_j\rangle).$$

This definition is independent of the choice of orthonormal bases.

Let G/K be a Riemannian homogeneous space. We denote by o the origin of G/K. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$ we define $\sigma_K(V,W)$, the angle between V and W, by

$$\sigma_K(V,W) = \int_K \sigma((g_x)_*^{-1}V,\ k_*^{-1}(g_y)_*^{-1}W)d\mu_K(k),$$

where g_x and g_y are elements of G with $g_x o = x$ and $g_y o = y$. We note that this definition is independent of the choice of g_x and g_y.

With these notations, the generalized Poincaré formula for homogeneous spaces can be stated as follows:

Theorem 2.1 (Howard [8])**.** *Suppose that G is unimodular and let M and N be submanifolds of G/K with $\dim M + \dim N \geq \dim(G/K)$. Then we have*

$$\int_G \mathrm{vol}(M \cap gN)d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^{\perp}M, T_y^{\perp}N)d\mu_{M\times N}(x,y). \tag{2.1}$$

Let V_o be a p-dimensional subspace of $T_o(G/K)$. Then a p-dimensional submanifold M of G/K is said to be *of type* V_o if for each x in M there exists g_x in G with $(g_x)_*^{-1}(T_xM) = V_o$. Equality (2.1) implies that if M is a submanifold of G/K of type V_o and N of type W_o for some subspaces V_o and W_o of $T_o(G/K)$, then σ_K is a constant function on $M \times N$ and

$$\int_G \mathrm{vol}(M \cap gN)d\mu_G(g) = \sigma_K(V_o^{\perp}, W_o^{\perp})\mathrm{vol}(M)\mathrm{vol}(N). \tag{2.2}$$

It is clear that if G/K is a real space form then all $(\dim V_o)$-dimensional submanifolds are of type V_o. So (2.2) yields the Poincaré formula in real space forms. In the case where G/K is a complex projective space $\mathbb{C}P^n$, any p-dimensional complex submanifolds is a type V_o for any p-dimensional complex subspace V_o in $T_o(G/K)$. Thus for any complex submanifolds M and N of complex dimensions p and q with $p+q \geq n$, we have

$$\begin{aligned}\int_{U(n+1)} &\mathrm{vol}(M \cap gN)d\mu_{U(n+1)}(g) \\ &= \frac{\mathrm{vol}(\mathbb{C}P^{p+q-n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{C}P^p)\mathrm{vol}(\mathbb{C}P^q)}\mathrm{vol}(M)\mathrm{vol}(N).\end{aligned} \tag{2.3}$$

This was first obtained by Santaló [21]. For further details, see [8] as a reference.

By the representation theory of compact Lie groups, we can show that σ_K is constant for complex submanifolds in an almost Hermitian homogeneous space under some assumptions. Hence Poincaré formula can be expressed as a constant times of volumes of two submanifolds. More precisely,

Proposition 2.2 ([11], [20]). *Let G be a unimodular Lie group and G/K an almost Hermitian homogeneous space of complex dimension n. Assume that K acts irreducibly on the exterior algebras $\wedge^p(T_o(G/K))^{(1,0)}$ and $\wedge^q(T_o(G/K))^{(1,0)}$ with $p+q \leq n$. Then there exists a positive constant C such that for any almost complex submanifolds M and N of G/K of complex dimensions $(n-p)$ and $(n-q)$ respectively*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = C \mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

Theorem 2.3 ([20]). *Let G/K be an irreducible Hermitian symmetric space of complex dimension n. Assume that K acts irreducibly on the exterior algebra $\wedge^p(T_o(G/K))^{(1,0)}$. Then for any complex submanifolds M and N of G/K of complex dimensions $(n-p)$ and $(n-q)$ respectively with $p+q \leq n$, we have*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{(n-p)!(n-q)!\mathrm{vol}(K)}{n!(n-p-q)!}\mathrm{vol}(M)\mathrm{vol}(N).$$

Remark 2.4. If $G/K = \mathbb{C}P^n$ and $G = U(n+1)$, then K-action on $T_o(G/K)$ is orbit equivalent to the canonical action of a unitary group $U(n)$. So K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$ for any p. Thus Theorem 2.3 is actually an extension of the formula (2.3) due to Santaló.

In Table 1, we give p when K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$ for irreducible Hermitian symmetric spaces. Although we show the case of compact type, it is clear that their non-compact duals also give the same results of Table 1.

	compact type	
A III	$SU(l)/S(U(m) \times U(l-m))$	any p (if $m=1$)
		p=1 (if $m \geq 2$)
D III	$SO(2l)/U(l)$	$p=1,2$
BD I	$SO(2l)/SO(2) \times SO(2l-2)$	$p \neq l-1$
	$SO(2l+1)/SO(2) \times SO(2l-1)$	any p
C I	$Sp(l)/U(l)$	$p=1,2$
E III	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10)+\mathbb{R})$	$p=1,2,3$
E VII	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6+\mathbb{R})$	$p=1,2,3,4$

Table 1

3. Kinematic formulas in real space forms

Here we shall review some definitions and fundamental properties with respect to the kinematic formula in Reimannian homogeneous spaces obtained by Howard [8]. And we determine the concrete forms of kinematic formulas of degree 2 in real space forms, completely.

Let V_o be a linear subspace of $T_o(G/K)$. We define a vector space $\mathrm{II}(V_o)$ to be

$$\mathrm{II}(V_o) = \left\{h \mid h : V_o \times V_o \to V_o^{\perp}; \text{symmetric bilinear}\right\},$$

where $V_o^{\perp}$ is the normal space of V_o in $T_o(G/K)$. An element $h \in \mathrm{II}(V_o)$ can be thought of as the second fundamental form of a submanifold of G/K which pass through o and have V_o as the tangent space at o. Let $K(V_o)$ be the stabilizer of V_o in K, that is, $K(V_o) = \{k \in K \mid k_* V_o = V_o\}$. This group $K(V_o)$ acts on $\mathrm{II}(V_o)$ in the following manner:

$$(kh)(u, v) = k_* h(k_*^{-1}u, k_*^{-1}v) \qquad (u, v \in V_o) \tag{3.1}$$

for $k \in K(V_o)$ and $h \in \mathrm{II}(V_o)$. Here we may consider a polynomial $\mathcal{P}$ on the vector space $\mathrm{II}(V_o)$ which is invariant under $K(V_o)$, that is, $\mathcal{P}(kh) = \mathcal{P}(h)$ for all $k \in K(V_o)$ and $h \in \mathrm{II}(V_o)$. In addition, let M be a submanifold of G/K of type V_o. Then for each $x \in M$ there exists $g_x \in G$ such that $(g_x)_*^{-1}(T_x M) = V_o$. Thus $g_x^{-1}M$ is a submanifold of G/K which pass through o with $T_o(g_x^{-1}M) = V_o$. For the second fundamental form h_x^M of M at $x \in M$, we define

$$\mathcal{P}(h_x^M) = \mathcal{P}(h_o^{g_x^{-1}M}).$$

It is easy to check that this definition is independent of the choice of $g_x \in G$ by the invariance of $\mathcal{P}$ under $K(V_o)$. Now we can define the integral invariant of M with respect to a polinomial $\mathcal{P}$ by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h_x^M) d\mu_M. \tag{3.2}$$

When $\mathcal{P}$ is an invariant homogeneous polynomial of degree l, we call $I^{\mathcal{P}}(M)$ an integral invariant *of degree* l. We remark that $I^{\mathcal{P}}$ is invariant under G. Many of the invariants that are usually dealt with are of the form $I^{\mathcal{P}}$. For example, if $\mathcal{P} \equiv 1$ then $I^{\mathcal{P}}(M) = \mathrm{vol}(M)$. An integration of square of the mean curvature on M is called the Willmore functional.

We also define a vector space $\mathrm{EII}(T_o(G/K))$ as follows:

$$\mathrm{EII}(T_o(G/K)) = \{h \mid h : T_o(G/K) \times T_o(G/K) \to T_o(G/K); \text{symmetric bilinear}\}.$$

Since K acts on $\mathrm{EII}(T_o(G/K))$ as in (3.1), we can also define integral invariants from polynomials on $\mathrm{EII}(T_o(G/K))$ that are invariant under K in the same manner with (3.2).

With these preliminaries, the kinematic formula in Riemannian homogeneous spaces can now be stated as follows:

Theorem 3.1. (Howard [8] paragraph 4.10) *Let G/K be a Riemannian homogeneous space and assume that G is unimodular. Let V_o and W_o be linear subspaces of $T_o(G/K)$ with $\dim(V_o) + \dim(W_o) \geq \dim(G/K)$ and $\mathcal{P}$ a homogeneous polynomial of degree l on $\mathrm{EII}(T_o(G/K))$ which is invariant under K such that*

$$\int_K \sigma(V_o^{\perp}, k_* W_o^{\perp})^{1-l} d\mu_K(k) < \infty. \tag{3.3}$$

Then there exists a finite set of pairs $\{\mathcal{Q}_\alpha, \mathcal{R}_\alpha\}_{\alpha \in A}$ such that

(1) *each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(V_o)$ invariant under $K(V_o)$,*
(2) *each $\mathcal{R}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(W_o)$ invariant under $K(W_o)$,*
(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ *for each α,*

(4) *for all compact submanifolds (possibly with boundaries) M of type V_o and N of type W_o in G/K the kinematic formula*

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_{\alpha} I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N) \tag{3.4}$$

holds.

Remark 3.2. The inequality (3.3) is the convergence condition for the integral. If G/K is a real space form and G is the group either of all isometries or of all orientation preserving isometries of G/K, then the condition (3.3) can be replaced by the manageable inequality $l \leq \dim(M) + \dim(N) - \dim(G/K) + 1$.

In his proof of Theorem 3.1 the integration on G was reduced to that on K. From this fact we arrive at the "*transfer principle*", a method of transferring kinematic formulas from one homegeneous space to any other homogeneous space with the same isotropy subgroup. Refer to [8] for detailed discussions of these facts.

Let us consider the case where G is the group of all isometries of a real space form. Then we can identify K with $O(T_o(G/K))$, the orthogonal group of $T_o(G/K)$, via the isomorphism of K to $O(T_o(G/K))$ with $a \mapsto a_*$. It is clear that if V_o is a p-dimensional subspace of $T_o(G/K)$ then $K(V_o) = O(V_o) \times O(V_o^\perp)$. If $\deg \mathcal{P} = 0$ then (3.4) is entirely Poincaré's formula. Here, we note that there are no homogeneous polynomials of odd degree on $\mathrm{II}(V_o)$ invariant under $K(V_o)$. The following polynomials $\mathcal{W}_{2l}$ are well-known invariant homogeneous polynomials on $\mathrm{II}(V_o)$ of degree $2l$.

$$\mathcal{W}_{2l}(h) = \sum \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \cdots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \cdots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \cdots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \cdots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix},$$

where the summation is over $1 \leq i_1, i_2, \dots, i_{2l} \leq p$ and $p+1 \leq k_1, k_2, \dots, k_l \leq n$. We note that the (a, b)-component of above matrix is $h^{k_{[(a+1)/2]}}_{i_a i_b}$, where $[x]$ means the greatest integer $[x]$ not greater than x. It is remark that these polynomials are characterized as the invariant polynomials which vanish identically on the second fundamental forms of some generalized cylinders. For each submanifold M of G/K, we introduce the integral invariants $\mu_{2l}(M)$ defined by

$$\mu_{2l}(M) = I^{\mathcal{W}_{2l}}(M).$$

Indeed, $\mathcal{W}_{2l}$ can be also expressed as a homogeneous polynomial of the components of the curvature tensor of M from Gauss formula, so $\mu_{2l}(M)$ is an intrinsic invariant of M. For these integral invariants μ_{2l}, the Cheren-Federer kinematic formula (1.2) holds. In fact, this formula holds in any real space forms by the transfer principle. The value of the constants $a(p, q, n, k, l)$ were first computed by Chern [4]. Afterward Nijenhuis [15] gave a simpler expression.

The space of homogeneous polynomials of degree 2 invariant under $K(V_o)$ is spanned by the two polynomials

$$\mathcal{Q}_1(h) = \sum_{i,j,k} \left(h^k_{ij}\right)^2, \qquad \mathcal{Q}_2(h) = \sum_k \left(\sum_i h^k_{ii}\right)^2,$$

where $1 \leq i, j \leq p,\ p+1 \leq k \leq n$. And if $2 \leq p \leq n-1$ these two polynomials are independent. Geometrically, $\mathcal{Q}_1(h)$ is the square of the norm of the second fundamental form, and $\mathcal{Q}_2(h)$ is p^2 times the square of the mean curvature. However, it is convenient for us to take the basis

$$\mathcal{W}_2 = \mathcal{Q}_2 - \mathcal{Q}_1, \qquad \mathcal{U}_p = p\mathcal{Q}_1 - \mathcal{Q}_2.$$

In the case of the integral invariant of degree 2, the Chern-Federer formula (1.2) can be stated as follows:

Proposition 3.3. *Assume that $2 \leq p+q-n$. Then there exist constants $a(p,q,n)$ so that for any compact submanifolds M^p and N^q of a real space from G/K the kinematic formula*

$$\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) = a(p,q,n) I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + a(q,p,n)\mathrm{vol}(M) I^{\mathcal{W}_2}(N)$$

holds.

On the other hand, $\mathcal{U}_p$ is characterized as the invariant polynomial which vanish identically on the second fundamental forms of p-dimensional sphere in $\mathbb{R}^n$. Hence we easily have the following:

Proposition 3.4. (Howard [8] paragraph 8.5) *Assume that $2 \leq p+q-n$. Then there exist constants $b(p,q,n)$ so that for any compact submanifolds M^p and N^q of a real space from G/K the kinematic formula*

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN) d\mu_G(g) = b(p,q,n) I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + b(q,p,n)\mathrm{vol}(M) I^{\mathcal{U}_q}(N)$$

holds.

The space of invariant homogeneous polynomials of degree 2 on $\mathrm{II}(V_o)$ is spanned by $\mathcal{W}_2$ and $\mathcal{U}_p$. Hence if we can give explicitly the constants $a(p,q,n)$ and $b(p,q,n)$ appearing in Propositions 3.3 and 3.4, then we can obtain the concrete form of kinematic formulas of this case completely.

Theorem 3.5 ([10]). *In Propositions 3.3 and 3.4*

$$\begin{aligned} a(p,q,n) &= \frac{2(p+q-n-1)}{p-1} \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}, \\ b(p,q,n) &= \frac{2(p+q-n+2)(p+q-n-1)}{(p+2)(p-1)} \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}. \end{aligned}$$

In the previous Theorem we investigated the kinematic formulas for the integral invariants of degree 2 under the condition $2 \leq p+q-n$. However, it remains the case of $p+q-n=1$, that is, the intersection becomes a curve. In this case the polynomial $\mathcal{Q}_1$ coincides with $\mathcal{Q}_2$ and these polynomials are the square of the curvature of the curve.

Let M^p and N^{n-p+1} be compact submanifolds of a real space form G/K of dimension n. Then the intersection $M \cap gN$ becomes a curve for almost all $g \in G$. Here we take the total square curvature of $M \cap gN$ as an integral invariant of the curve, and denote

$$I^{\kappa^2}(M \cap gN) = \int_{M \cap gN} \kappa^2 ds.$$

From Theorem 3.1, we have the following:

Proposition 3.6. *There exist constants $c(p,n)$ and $d(p,n)$ such that*

$$\begin{aligned}&\int_G I^{\kappa^2}(M\cap gN)d\mu_G(g)\\ =\ &\left(c(p,n)I^{\mathcal{W}_2}(M)+d(p,n)I^{\mathcal{U}_p}(M)\right)\mathrm{vol}(N)\\ &+\mathrm{vol}(M)\left(c(n-p+1,n)I^{\mathcal{W}_2}(N)+d(n-p+1,n)I^{\mathcal{U}_{n-p+1}}(N)\right).\end{aligned}$$

We can give the explicit value of $c(p,n)$ and $d(p,n)$.

Theorem 3.7 ([10])**.** *In Proposition 3.6,*

$$\begin{aligned}c(p,n) &= \frac{4\pi}{p-1}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-p+1})},\\ d(p,n) &= \frac{12\pi}{(p+2)(p-1)}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-p+1})}.\end{aligned}$$

4. Hamiltonian volume minimizing properties of minimal Lagrangian submanifolds

In the previous sections, we studied the formuation of kinematic formulas. Here we attempt to apply the Poincaré formula to show Hamiltonian volume minimizing properties of minimal Lagrangian submanifolds in Kähler manifolds.

Let (M,ω) be a $2n$-dimensional closed symplectic manifold with symplectic form ω and L be an n-dimensional closed submanifold of M. Then L is said to be *Lagrangian* if $\omega|_{TL}\equiv 0$. Hamiltonian isotopies of (M,ω) are defined as follows. If a smooth function $F: M\times[0,1]\to\mathbb{R}$ is given, then we can uniquely define the vector field X_t on M for each $t\in[0,1]$ such that

$$\omega(X_t,\cdot)=dF(\cdot,t).$$

Therefore, we have the flow $\{\phi_t\}_{t\in[0,1]}$ of diffeomorphisms on M defined by the differential equation

$$\frac{d}{dt}\phi_t(x)=X_t(\phi_t(x))$$

with initial condition $\phi_0=id_M$. The time 1-map ϕ_1 of this flow is called a *Hamiltonian diffeomorphism.* The set of all Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M,\omega)$. We can check that $\mathrm{Ham}(M,\omega)$ is a subgroup of the identity component $\mathrm{Diff}_0(M)$ of the diffeomorphism group of M. By definition, a Hamiltonian diffeomorphism ϕ of M preserves the symplectic structure (*i.e.*, $\phi^*\omega=\omega$). Therefore, if L is a Lagrangian submanifold of M, then $\phi(L)$ is also Lagrangian.

Hereafter we restrict our attention to Kähler manifolds to introduce the volume functional. Let (M,ω,J) be a closed connected Kähler manifold, and L its Lagrangian submanifold. We set $\mathrm{Ham}(L)=\{\phi(L)\mid\phi\in\mathrm{Ham}(M,\omega)\}$. If the volume functional on $\mathrm{Ham}(L)$ is stationary at L, then L is called *Hamiltonian minimal.* A Hamiltonian minimal Lagrangian submanifold is said to be *Hamiltonian stable* if any second variation of the volume functional at L on $\mathrm{Ham}(L)$ is non-negative. Moreover, if L has the least volume in $\mathrm{Ham}(L)$, then L is said to be *Hamiltonian volume minimizing.* These notions are first introduced by Y. G. Oh. In his paper [16], he studied the second variational formula for minimal Lagrangian submanifolds in Kähler manifolds. And he showed a following local criterion.

Proposition 4.1 (Oh)**.** *Let (M, ω, J) is a Kähler-Einstein manifold with Einstein constant c and L be its minimal Lagrangian submanifold. Then L is Hamiltonian stable if and only if $\lambda_1 \geq c$, where λ_1 denotes the first eigenvalue of Laplace-Beltrami operator $-\Delta$ acting on $C^\infty(M)$.*

This says that local Hamiltonian stabilities of minimal Lagrangian submanifolds of Kähler-Einstein manifolds are only depend on their properties in Riemannian geoemtry, namely first eigenvalue of Laplacian. From this proposition, we can have many Hamiltonian stable minimal Lagrangian submanifolds (See [1] and [16] for references).

On the other hand, we have few examples of Hamiltonian volume minimizing Lagrangian submanifolds. Trivial examples are *special Lagrangian submanifolds* in Ricci-flat Kähler manifolds. In fact, they are calibrated submanifolds and have minimun volume in their homology classes. But, in general, it is difficult to check whether a minimal Lagrangian submanifold L is Hamiltonian volume minimizing or not, when L is not a calibrated submanifold. At the end of his paper, Y.G.Oh posed a following conjecture:

Conjecture 4.2 (Oh)**.** *Let M be a Kähler-Einstein manifold with an involutive anti-holomorphic isometry τ. Suppose that the fixed point set of τ*

$$L := \mathrm{Fix}\ \tau$$

is also a compact Einstein manifold with positive Ricci curvature. Then L is Hamiltonian volume minimizing.

For this conjecture, Kleiner and Oh gave only one non-trivial example $\mathbb{R}P^n \subset \mathbb{C}P^n$. Recently, Goldstein [7] obtained stronger statement for the volume minimality of real projective spaces in $\mathbb{C}P^n$.

Now we set up our problem for Hamiltonian volume minimizing properties of Lagrangian submanifolds as following.

Problem 4.3. *Does a real form of a Hermitian symmetric space of compact type which is Hamiltonian stable have Hamiltonian volume minimizing property or not?*

Hermitian symmetric spaces of compact type are important examples of Kähler-Einstein manifolds with positive Ricci curvature. A *real form* is a totally geodesic Lagrangian submanifold defined by the fixed point set of an involutive anti-holomorphic isometry in a Hermitian symmetric space. Hamiltonian stabilities of real forms were already determined completely by Amarzaya and Ohnita [1]. So we are interested in their Hamiltonian volume minimizing properties. The reason why we set the problem like this is that we want to apply the following Arnold-Givental inequality in the Lagrangian intersection theory. That is a global investigation in symplectic geometry.

Theorem 4.4 (Oh [17], [18] and [19])**.** *Let (M, ω) be a compact symplectic manifold such that there exists an integrable almost complex structure J for which the triple (M, ω, J) becomes a compact Hermitian symmetric space. Let $L = \mathrm{Fix}\ \tau$ be the fixed point set of an anti-holomorphic involutive isometry τ on M. Assume that the minimal Maslov number of L is greater than or equal to 2. Then for any Hamiltonian diffeomorphism ϕ of M such that L and $\phi(L)$ intersect transversely,* *the inequality*

$$\sharp(L \cap \phi(L)) \geq \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) \tag{4.1}$$

holds.

Now we state our conjecture.

Conjecture 4.5 (IOS)**.** *Let $(G/K, \omega, J)$ be a Hermitian symmetric space of compact type and $L = \mathrm{Fix}\ \tau$ its real form. If L is Hamiltonian stable, then we have*

$$C\mathrm{vol}(L)\mathrm{vol}(N) \geq \int_G \sharp(L \cap gN) d\mu_G(g)$$

where

$$C = \frac{(\sum \mathrm{rank} H_*(L; \mathbb{Z}_2))\mathrm{vol}(G)}{\mathrm{vol}(L)^2}$$

for any Lagrangian submanifold N.

The assumption that L is Hamiltonian stable is, of course, a necessary condition for L to be Hamiltonian volume minimizing.

Proposition 4.6. *Under the same assumption as Conjecture 4.5, if Conjecture 4.5 is true, then the totally geodesic Lagrangian submanifold $L = \mathrm{Fix}\ \tau$ whose minimal Maslov number is greater than or equal to 2 in $(G/K, \omega, J)$ is Hamiltonian volume minimizing.*

Proof. By Theorem 4.4 and Conjecture 4.5, we have

$$\begin{aligned}
C\mathrm{vol}(L)\mathrm{vol}(\phi(L)) &\geq \int_G \sharp(L \cap g \circ \phi(L)) d\mu_G(g) \\
&\geq \int_G \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) d\mu_G(g) \\
&= \mathrm{vol}(G) \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) \\
&= C\mathrm{vol}(L)^2.
\end{aligned}$$

Hence,

$$\mathrm{vol}(\phi(L)) \geq \mathrm{vol}(L).$$

□

Remark 4.7. The method we use here, a combination of Lagrangian intersection theorems in symplectic geometry and Poincaré formulas in integral geometry, was first pointed out by Kleiner and Oh.

Now we shall discuss the above conjecture in the case $S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$.

Let G/K be a Riemannian homogeneous space. The linear isotropy representation induces an action of K on the Grassmannian manifold $G_p(T_o(G/K))$ consisting of all p-dimensional subspaces in the tangent space $T_o(G/K)$ at o in a natural way. Although $\sigma_K(T_x^\perp L, T_y^\perp N)$ is defined as an integral on K, we can consider that it is defined as an integral on an orbit of K-action on the Grassmannian manifold. So

$\sigma_K(\cdot,\cdot)$ can be regarded as a function defined on the product of the orbit spaces of such K-actions. In the case where G/K is a real space form, $\sigma_K(T_x^\perp L, T_y^\perp N)$ is constant since K acts transitively on the Grassmannian manifold. This implies that the Poincaré formula is expressed as a constant times of the product of the volumes of N and L. In general, such K-actions are not transitive. However, if we can define an invariant for orbits of this action, which is called an *isotropy invariant*, then using this we can express the Poincaré formula explicitly.

Next, we define isotropy invariants for surfaces in $S^2 \times S^2$ and give an explicit Poincaré formula for its Lagrangian surfaces.

Let G be the identity component of the isometry group of $S^2 \times S^2$, that is, $G = SO(3) \times SO(3)$. Then the isotropy group K at $o = (p_1, p_2)$ in $S^2 \times S^2$ is isomorphic to $SO(2) \times SO(2)$, and $S^2 \times S^2$ can be identified with a coset space G/K. Assume thet G is equipped with an invariant metric normalized so that G/K becomes isometric to the product of unit spheres. We decompose the tangent space $T_o(G/K)$ as

$$T_o(G/K) = T_{p_1}(S^2) \oplus T_{p_2}(S^2).$$

We consider the oriented 2-plane Grassmannian manifold $\tilde{G}_2(T_o(G/K))$. Take an origin $V_o := T_{p_1}(S^2)$ and express $\tilde{G}_2(T_o(G/K))$ as a coset space

$$\tilde{G}_2(T_o(G/K)) = SO(4)/(SO(2) \times SO(2)) =: G'/K'.$$

Now we study the K-action on $\tilde{G}_2(T_o(G/K))$ and define isotropy invariants. In this case the actions of K and K' on $\tilde{G}_2(T_o(G/K))$ are equivalent by $\mathrm{Ad} : K \to K'$. Therefore it suffices to consider the orbit space of the isotropy action of $\tilde{G}_2(T_o(G/K))$. It is well known that the orbit space of the isotropy action of a symmetric space of compact type can be identified with a fundamental cell of a maximal torus. Hence we can define the isotropy invariant by a coordinate of a maximal torus. We denote by $\mathfrak{g}'$ and $\mathfrak{k}'$ the Lie algebra of G' and K', respectively. Then we have a canonical direct sum decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$, where

$$\mathfrak{m}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X \in M_2(\mathbb{R}) \right\}.$$

We take a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{m}'$ as follows:

$$\mathfrak{a}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \theta_1, \theta_2 \in \mathbb{R} \right\}.$$

Then the set of positive restricted roots of $(\mathfrak{g}', \mathfrak{k}')$ with respect to $\mathfrak{a}'$ is

$$\Delta = \{\theta_1 + \theta_2, \theta_1 - \theta_2\}.$$

So we have a fundamental cell C of $\mathfrak{a}'$:

$$C = \left\{ Y = \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \begin{array}{l} 0 \le \theta_1 + \theta_2 \le \pi \\ 0 \le \theta_1 - \theta_2 \le \pi \end{array} \right\}.$$

Thus the isotropy invariants in this case are given by $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$. $S^2 \times S^2$ has a Kähler form $\omega_0 \oplus \omega_0$, where ω_0 denotes the standard Kähler form of $S^2 \cong \mathbb{C}P^1$. It is easy to see that the geometric meaning of $\theta_1 - \theta_2$ is the Kähler angle of 2-dimensional subspace $\mathrm{Exp}Y$ of $T_o(G/K)$. On the other hand, $S^2 \times S^2$ has another Kähler structure $\omega_0 \oplus (-\omega_0)$. We can also check that $\theta_1 + \theta_2$ is the Kähler angle of $\mathrm{Exp}Y$ with respect to $\omega_0 \oplus (-\omega_0)$.

Using these isotropy invariants we obtain the following formula from Theorem 2.1.

Theorem 4.8 (IOS [9]). *Let N and L be Lagrangian surfaces in $S^2 \times S^2$. We assume that L is a product of curves in S^2. Then we have*

$$\int_G \sharp(L \cap gN) d\mu_G(g) = 4\mathrm{vol}(L) \int_N \mathrm{length}(\mathrm{Ellip}(\sin^2\theta_x, \cos^2\theta_x)) d\mu_N(x),$$

where $2\theta_x - \pi/2$ is the Kähler angle of $T_x^\perp N$ with respect to $\omega_0 \oplus (-\omega_0)$ and $\mathrm{Ellip}(\alpha, \beta)$ *denotes an ellipse defined by*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Theorem 4.8 yields the following immediately.

Corollary 4.9. *Let N and L be surfaces of $S^2 \times S^2$. Suppose that N is Lagrangian and L is a product of curves in S^2. Then the following inequality holds*:

$$\int_{SO(3)\times SO(3)} \sharp(L \cap gN) d\mu_G(g) \leq 16\mathrm{vol}(N)\mathrm{vol}(L). \tag{4.2}$$

Moreover, the equality holds if and only if the Lagrangian surface N is also a product of curves in S^2.

Let $L := S^1(1) \times S^1(1)$, then the mimimal Maslov number of L is 2 and

$$\frac{\sum_{i=0}^{2} \mathrm{rank} H_i(L, \mathbb{Z}_2) \mathrm{vol}(SO(3) \times SO(3))}{\mathrm{vol}(L)^2} = \frac{4 \cdot (8\pi^2 \cdot 8\pi^2)}{(4\pi^2)^2} = 16.$$

Thus we have the conclusion by Proposition 4.6:

Theorem 4.10 (IOS [9]). *Let $L := S^1(1) \times S^1(1)$ be a totally geodesic Lagrangian torus in $(S^2(1) \times S^2(1), \omega_0 \oplus \omega_0)$. Then for any Hamiltonian diffeomorphism $\phi \in$* $\mathrm{Ham}(S^2 \times S^2)$, *we have*

$$\mathrm{vol}(\phi(L)) \geq \mathrm{vol}(L).$$

Moreover, if $\mathrm{vol}(\phi(L)) = \mathrm{vol}(L)$ *holds, then there exists an isometry g of $S^2(1) \times S^2(1)$ such that $\phi(L) = gL$.*

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