弱鏡映部分多様体とその周辺の話題

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Minimal submanifolds

$f : M \longrightarrow (\widetilde{M}, \langle \cdot, \cdot \rangle)$ immersion

$\mathrm{Vol} : \mathrm{Imm}(M, \widetilde{M}) \longrightarrow \mathbb{R}$ volume functional

$F : M \times (-\varepsilon, \varepsilon) \longrightarrow \widetilde{M}$, compact support, $F(\cdot, 0) = f$

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(F(M,t)) = -\int_M \langle \mathbf{H}, V \rangle d\mu_M$$

where $\mathbf{H}$: mean curvature vector field

$V := F_* \frac{\partial}{\partial t}$: variational vector field

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(F(M,t)) = 0 \text{ for } \forall F \iff \mathbf{H} \equiv 0$$

minimal submanifold

Mean curvature vector fields

$M \subset (\widetilde{M}, \langle\cdot,\cdot\rangle, \widetilde{\nabla})$

$$T_x\widetilde{M} = T_xM \oplus T_x^{\perp}M$$

For $X, Y \in \Gamma(TM), \quad \xi \in \Gamma(T^{\perp}M)$

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad \text{(Gauss formula)}$$

h : second fundamental form

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^{\perp}_X \xi \qquad \text{(Weingarten formula)}$$

A_ξ : shape operator

$$\mathbf{H} := \mathrm{trace}(h) \in \Gamma(T^{\perp}M)$$

$$\langle h(X,Y), \xi\rangle = \langle Y, A_\xi(X)\rangle$$

Definition of reflective submanifold

Definition (D. Leung)

A connected component M of the fixed point set of an involutive isometry σ of $(\widetilde{M}, \langle\cdot,\cdot\rangle)$ is called a **reflective submanifold**.

- A reflective submanifold is a totally geodesic submanifold.
- For all $x \in M$ and all $\xi \in T_x^\perp M$, the reflection σ satisfies

$$\sigma(x) = x, \qquad (d\sigma)_x \xi = -\xi, \qquad \sigma(M) = M.$$

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Definition of a weakly reflective submanifold

Definition

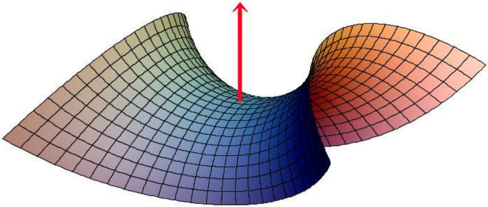
$M \subset \widetilde{M}$: **weakly reflective submanifold (WRS)**

$\overset{\text{def}}{\Longleftrightarrow}$ for each $x \in M$ and each $\xi \in T_x^{\perp} M$,

there exists an isometry σ_ξ of $\widetilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M.$$

We call σ_ξ a **reflection** of M with respect to ξ.



An example of a weakly reflective submanifold

Example

$$M = S^{n-1}(1) \times S^{n-1}(1) = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid u, v \in S^{n-1}(1)\}$$

is a weakly reflective submanifold in $S^{2n-1}(\sqrt{2})$.

$x = (e_1, e_1) \in M$
$\xi = (e_1, -e_1) \in T_x^{\perp} M$

$$\begin{array}{rcl} \sigma_\xi : S^{2n-1}(\sqrt{2}) & \longrightarrow & S^{2n-1}(\sqrt{2}) \\ (u,v) & \longmapsto & (v,u) \end{array}$$

Then σ_ξ satisfies

$$\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(M) = M$$

Definition of austere submanifold

Definition (Harvey-Lawson)

$M \subset \widetilde{M}$: **austere submanifold**

$\overset{\text{def}}{\Longleftrightarrow}$ *for all* $\xi \in T_x^{\perp}M$, *the set of eigenvalues with their multiplicities of the shape operator* A_ξ *of* M *is invariant under the multiplication by* -1.

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

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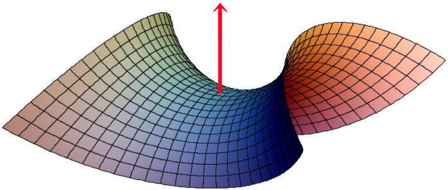
Classes of submanifolds

Proposition

reflective $\subset$ *WRS* $\subset$ *austere* $\subset$ *minimal*



$$(d\sigma_\xi)_x^{-1} A_\xi (d\sigma_\xi)_x = -A_\xi$$

Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeniety one action on a Riemannian manifold is a weakly reflective submanifold.

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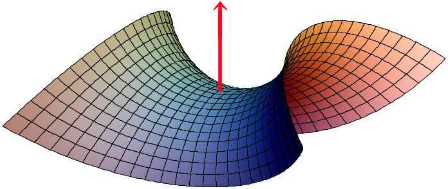

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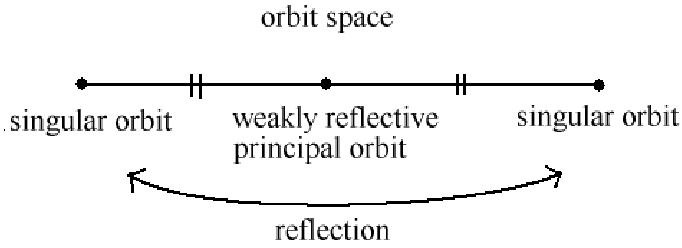
Any singular orbit of a cohomogeniety one action on a Riemannian manifold is a weakly reflective submanifold.

Orbits of cohomogeneity one actions

Proposition

Let $G \curvearrowright (\widetilde{M}, \langle \cdot, \cdot \rangle)$ be a cohomogeneity one action with two singular orbits. If $\exists$ principal orbit which is a WRS in $\widetilde{M}$, then

1. *it has a same distance from two singular orbits,*
2. *two singular orbits are isometric.*



Orbits of s-representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

$$K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{m} \cong T_o(G/K) : \ s\text{-representation}$$

For $H \in S \subset \mathfrak{m}$, $\quad \mathrm{Ad}(K)H \subset S$: R-space

$\mathfrak{a} \subset \mathfrak{m}$: maximal abelian subspace

$$\mathfrak{m}_\lambda := \{X \in \mathfrak{m} \mid [H,[H,X]] = -\langle \lambda, H \rangle^2 X \ ({}^\forall H \in \mathfrak{a})\}$$

$R := \{\lambda \in \mathfrak{a} \mid \mathfrak{m}_\lambda \neq \{0\}\}$: restricted root system

$$\boxed{\mathfrak{m} = \sum_{\lambda \in R_+} \mathfrak{m}_\lambda}$$

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Orbits of s-representations

F : fundamental system of R

$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle > 0 \ (\forall \alpha \in F)\}$$: Weyl chamber

$$\mathrm{Ad}(K)\bar{C} = \mathfrak{m} \quad \rightsquigarrow \quad \boxed{\bar{C} \cap S \ : \text{orbit space of } K \curvearrowright S}$$

For $\Delta \subset F$

$$C^{\Delta} = \left\{ H \in \bar{C} \;\middle|\; \begin{array}{l} \langle \alpha, H\rangle > 0 \ (\alpha \in \Delta), \\ \langle \beta, H\rangle = 0 \ (\beta \in F - \Delta) \end{array} \right\}$$

$$\bar{C} \cap S = \bigcup_{\Delta \subset F} (C^{\Delta} \cap S) \quad \text{disjoint union}$$

Theorem (Hirohashi-Song-Takagi-Tasaki)

For a subset $\Delta \subset F$ there exists unique $H \in C^{\Delta} \cap S$ such that $\mathrm{Ad}(K)H$ is a minimal submanifold in the hypersphere S.

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Austere orbits of s-representations (Ikawa-Tasaki-S.)

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is an austere submanifold in the hypersphere S is one of the following list:

1. An orbit through a restricted root
2. $R = A_2; \quad H = 2e_1 - e_2 - e_3, \ e_1 + e_2 - 2e_3$
3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$
4. $R = D_n; \quad H = e_1$
5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$
6. $R = B_2$ with constant multiplicities; $\quad H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$
7. $R = G_2; \quad H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$

Moreover, in the cases (1)∼(5), these austere orbits are weakly reflective submanifolds in S.

Gauss mapping

$f : M^l \longrightarrow S^n$ immersion

$$\begin{array}{rcl} \gamma : M & \longrightarrow & G_{l+1}(\mathbb{R}^{n+1}) \\ x & \longmapsto & \mathbb{R}f(x) \oplus T_{f(x)}(f(M)) \end{array}$$ Gauss map

Theorem (Ferus)

M^l : compact, connected manifold, $f : M \longrightarrow S^n$: immersion
Then, $\operatorname{rank}\gamma < F(l) \implies \operatorname{rank}\gamma = 0$

$F(l) = \min\{k \mid A(k) + k \geq l\}$: Ferus number
$A((2k+1)2^{c+4d}) = 2^c + 8d - 1$: Adams number

Problem (Ishikawa-Kimura-Miyaoka)

Is the inequality $\operatorname{rank}\gamma < F(l)$ *best possible for the implication* $\operatorname{rank}\gamma = 0$*?*

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Tangentially degenerate orbits

Theorem

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is tangentially degenerate is one of the following list:

1. *H : a long root*
2. *$R = G_2$; H : a short root*

Moreover, if λ is such a root, then

$$\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda.$$

- These orbits are weakly reflective submanifolds.
- In this list, we can find many orbits which satisfy the equality of the Ferus inequality.

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Tangentially degenerate orbits

In general, for the Gauss map γ of a submanifold $M \subset S^n$

$$\begin{aligned} \ker(d\gamma)_x &= \{X \in T_xM \mid h(X,Y) = 0, \ {}^\forall Y \in T_xM\} \\ &= \bigcap_{\xi \in T_x^\perp M} \ker(A_\xi) \end{aligned}$$

For the Gauss map γ of the orbit $\mathrm{Ad}(K)H$

$$\ker(d\gamma)_H = \bigcap_{k \in (Z_K^H)_0} \mathrm{Ad}(k) \sum_{\substack{\lambda \in R_+ \\ \lambda // H}} \mathfrak{m}_\lambda$$

where $Z_K^H = \{k \in K \mid \mathrm{Ad}(K)H = H\}$

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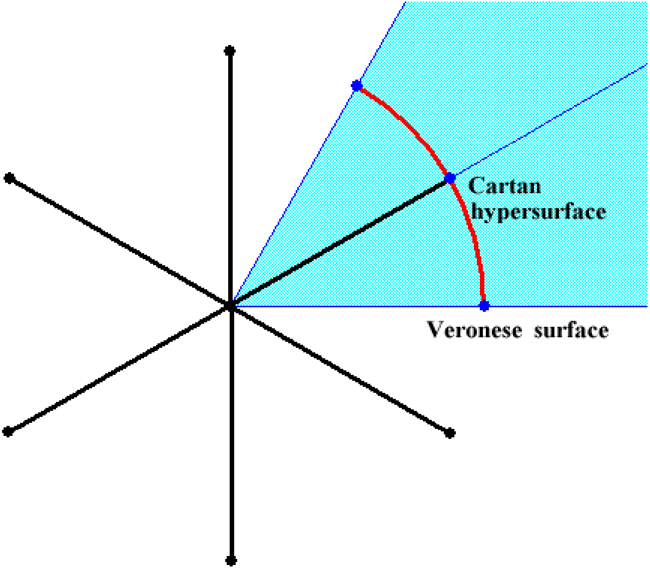
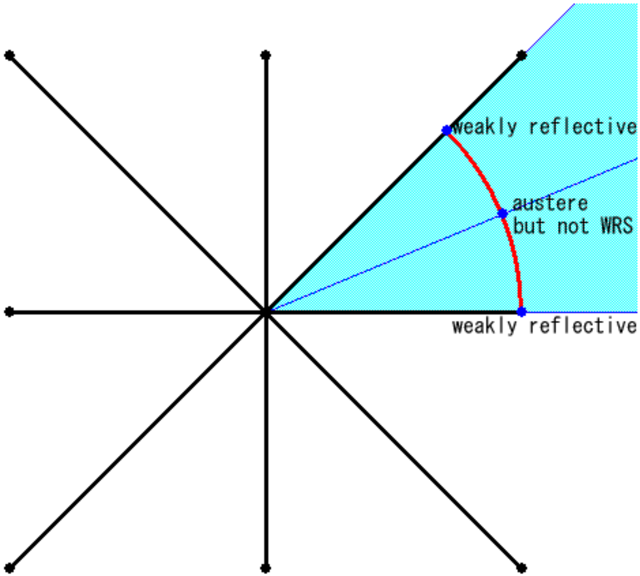
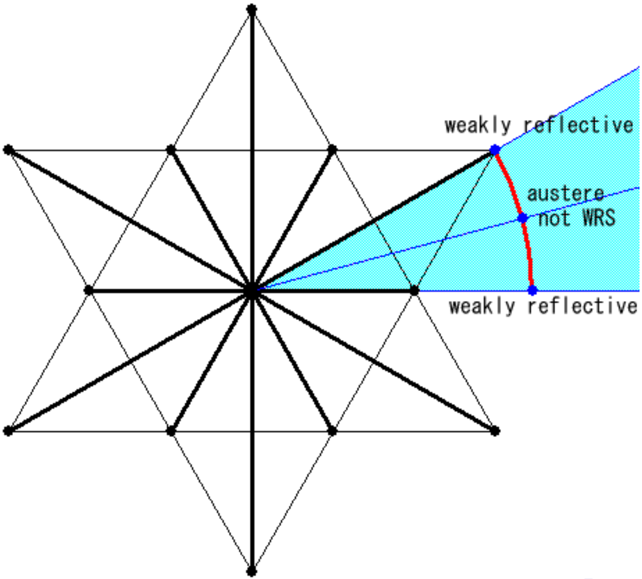
Case of type A_2



Case of type B_2



Case of type G_2



$(\widetilde{M}, J, \omega, \Omega)$: Calabi-Yau manifold

Definition

$L \subset \widetilde{M}$: **special Lagrangian submanifold** *of phase* θ

$\overset{\text{def}}{\Longleftrightarrow}$ L *is calibrated by* $\mathrm{Re}(e^{i\theta}\Omega)$ *for some* $\theta \in \mathbb{R}$.

$$\Longleftrightarrow \quad \begin{cases} \omega|_L \equiv 0 \\ \mathrm{Im}(e^{i\theta}\Omega|_L) \equiv 0 \end{cases}$$

Theorem (Harvey-Lawson)

A calibrated submanifold is volume minimizing in its homology class.

Special Lagrangian geometry

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Special Lagrangian normal bundles

$M \subset S^n$: submanifold

$$\begin{array}{rcll} \Phi : N^1 M \times S^1 & \longrightarrow & S^{2n+1} \subset \mathbf{R}^{2n+2} & \text{Legendrian} \\ (x,\ \xi,\ e^{i\theta}) & \longmapsto & (\cos\theta\, x,\ \sin\theta\, \xi) & \end{array}$$

Proposition (Harvey-Lawson)

$M \subset S^n$: *austere* $\iff$ Φ : *minimal*

Borrelli-Gorodski defined a map $\widetilde{\Phi}$ modifying Φ and showed that

A_ξ does not have 0-eigenvalue $\Longrightarrow$ $\widetilde{\Phi}$: Legendrian immersion

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Stenzel metric on T^*S^n

$$S^n = SO(n+1)/SO(n) =: G/K$$

$$T^*S^n = \left\{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi\rangle = 0\right\}$$

$$Q^n = \left\{(z_0,\dots,z_n) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=0}^{n} z_i^2 = 1\right\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$$

$$\begin{array}{rcll} \varphi : T^*S^n & \longrightarrow & Q^n \subset \mathbb{C}^{n+1} & \text{diffeomorphism} \\ (x,\xi) & \longmapsto & x\cosh(\|\xi\|) + \sqrt{-1}\dfrac{\xi}{\|\xi\|}\sinh(\|\xi\|) & \end{array}$$

- $G = SO(n+1)$ acts on T^*S^n and Q^n with cohomogeneity one.
- φ is G-equivariant.

Stenzel metric on T^*S^n

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Stenzel metric on T^*S^n

$$\omega_{Stz} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=0}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(r^2) dz_i \wedge d\bar{z}_j$$

where $r^2 = \|z\|^2$ and u is a real function given by
$U(\tau) = u(\cosh(\tau))$ where U satisfies

$$\frac{d}{d\tau}(U'(\tau))^n = cn(\sinh\tau)^{n-1} \qquad (c > 0).$$

Ω : holomorphic $(n,0)$-form on Q^n defined by

$$\Omega \wedge d(z_0^2 + z_1^2 + \cdots + z_n^2) = dz_0 \wedge dz_1 \wedge \ldots \wedge dz_n.$$

$(T^*S^n, J, \omega_{Stz}, \Omega)$ is a cohomogeneity one Calabi-Yau manifold.

Special Lagrangian normal bundles

$M \subset S^n$: submanifold

Then $L = N^*M$ is a Lagrangian submanifold of T^*S^n with respect to the canonical symplectic structure ω_0.

Theorem (Karigiannis-Min-Oo)

$L = N^*M$ is a Lagrangian submanifold of T^*S^n with respect to Stenzel metric ω_{Stz}.
Moreover, L is a special Lagrangian submanifold of T^*S^n if and only if M is an austere submanifold in S^n.

$\gamma(s) \subset \mathbb{C}$: regular curve

$p+q=n+1$

$$\Psi : I \times S^{p-1} \times S^{q-1} \longrightarrow Q^n \subset \mathbb{C}^n$$
$$(s,x,y) \longmapsto \left(\gamma(s)x_1,\ldots,\gamma(s)x_p, \sqrt{1-\gamma(s)^2}y_1,\ldots,\sqrt{1-\gamma(s)^2}y_q\right)$$

$L := \Psi(I \times S^{p-1} \times S^{q-1})$

$$\Longrightarrow \qquad \omega_{Stz}|_L \equiv 0$$

In addition, if γ satisfies

$$\mathrm{Im}\left(\gamma'\gamma^{p-1}(1-\gamma^2)^{\frac{q-2}{2}}\right) = 0,$$

$$\Longrightarrow \qquad \mathrm{Im}(\Omega|_L) \equiv 0$$

L is a special Lagrangian submanifold of $T^*S^n \cong Q^n$ w.r.t. ω_{Stz}.

Phase space

Case of $n = 6,\ p = 3,\ q = 4$

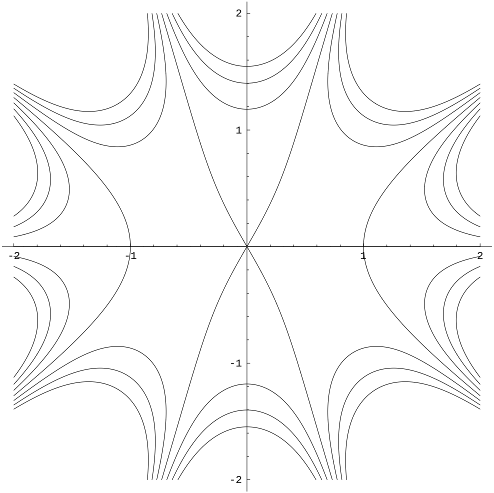

figure by 橋本 要

Further problems

- Classify all weakly reflective submanifolds in Riemannian symmetric spaces.
- Homogeneity of weakly reflective submanifolds.
  Construct examples of non-homogeneous weakly reflective submanifolds.
- Ishikawa-Kimura-Miyaoka's problem for the Ferus inequality.
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