Absolutely area-minimizing cones over some minimal submanifolds in S^n

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Variational problems for curves and surfaces and related topics
at Nara Women's University
July 1, 2009

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Methods to show area-minimizing property

- Geometric measure theory
- Calibration
- Area-nonincreasing retraction
- Integral geometry

Plan of this talk

1. Motivation and history of area-minimizing cones
2. Area-minimizing cones over some minimal orbits of s-representations
3. Weakly reflective submanifolds and austere submanifolds
4. Special Lagrangian cones

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Plateau problem

For given a Jordan curve Γ in $\mathbb{R}^n$, does there exist an area-minimizing surface with boundary Γ ?

Answer (Douglas and Rado)

YES! by geometric measure theory

But such a surface may have singularities

"integral current"

Theorem (Osserman)

An area-minimizing surface M^2 in $\mathbb{R}^3$ with smooth boundary has no singularities.

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Theorem (Almgren)

The dimension of the singular set of an area-minimizing submanifold M^k is equal or less than $k-2$.

A minimal surface M^2 in $\mathbb{R}^n$ $(n \geq 4)$ may have 0-dimensional singular set, i.e. isolated singularities.

Questions

- *What kind of singularities can appear on area-minimizing submanifolds?*
- *Study area-minimizing cones, which are tangent cones at isolated singularities.*

Introduction and Motivation

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Area-minimizing cones

$M^{k-1} \subset S^{n-1}(1) \subset \mathbb{R}^n$

$$\begin{aligned} C_M &= \{tx \mid x \in M,\ t \geq 0\} \subset \mathbb{R}^n \\ C_M^1 &= \{tx \mid x \in M,\ 0 \leq t \leq 1\} \end{aligned}$$

Definition

C_M is said to be **area-minimizing** *if C_M^1 has minimum volume among all submanifolds with boundary M.*

Proposition

$$M \subset S^{n-1} : \textit{minimal} \quad \Longrightarrow \quad C_M \subset \mathbb{R}^n : \textit{minimal}$$

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History and Examples of area-minimizing cones

1969 (Bombieri-DeGiorgi-Giusti) $M = S^n \times S^n \subset S^{2n+1}$ $(n \geq 3)$,
then $C_M \subset \mathbb{R}^{2n+2}$ is area-minimizing.

1972 (Lawson) $M = S^k \times S^l \subset S^{k+l+1}$ $(k + l \geq 7)$,
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1974 (Simoes)
$M = S^2 \times S^4 \subset S^7$, then $C_M \subset \mathbb{R}^8$ is area-minimizing.
$M = S^1 \times S^5 \subset S^7$, then $C_M \subset \mathbb{R}^8$ is NOT area-minimizing.

1991 (Lawlor) provided a criterion for a cone to be area-minimizing,
and classified all area-minimizing cones over product of spheres.

1988 (Cheng), 1994 (Kerckhove), 2000 (Hirohashi-Kanno-Tasaki),
2002 (Kanno)
Cones over some symmetric R-spaces are area-minimizing.

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Area-nonincreasing retraction

$$\Phi : \mathbb{R}^n \longrightarrow C_M \quad \text{differentiable retraction}$$

is **area-nonincreasing** if

$$J(d\Phi)_x = \sup\{\|(d\Phi)_x(u_1) \wedge \cdots \wedge (d\Phi)_x(u_k)\|\} \leq 1$$

for all $x \in \mathbb{R}^n$, where $u_1, \ldots u_k$ are orthonormal vectors of $T_x(\mathbb{R}^n)$

Proposition

If there exists an area-nonincreasing retraction $\Phi : \mathbb{R}^n \longrightarrow C_M$, then C_M is area-minimizing.

Proof. $$\mathrm{Vol}(C_M^1) \leq \mathrm{Vol}(\Phi(N)) \leq \mathrm{Vol}(N),$$

for any submanifold N with boundary M, since $C_M^1 \subset \Phi(N)$.

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Orbits of s-representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

$$K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{m} \cong T_o(G/K) : \ s\text{-representation}$$

For $H \in S \subset \mathfrak{m}$, $\quad \mathrm{Ad}(K)H \subset S$: R-space

$\mathfrak{a} \subset \mathfrak{m}$: maximal abelian subspace

R : restricted root system of $(\mathfrak{g}, \mathfrak{k})$

F : fundamental system of R

$$\mathcal{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\forall \alpha \in F)\} \quad \text{: Weyl chamber}$$

$$\mathrm{Ad}(K)\bar{\mathcal{C}} = \mathfrak{m} \quad \rightsquigarrow \quad \boxed{\bar{\mathcal{C}} \cap S \quad \text{: orbit space of } K \curvearrowright S}$$

Orbits of s-representations

For $\Delta \subset F$

$$\mathcal{C}^{\Delta} = \left\{ H \in \bar{\mathcal{C}} \;\middle|\; \begin{array}{l} \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \\ \langle \beta, H \rangle = 0 \ (\beta \in F - \Delta) \end{array} \right\}$$

$$\boxed{\bar{\mathcal{C}} \cap S = \bigcup_{\Delta \subset F} (\mathcal{C}^{\Delta} \cap S) \quad \text{disjoint union}}$$

Theorem (Hirohashi-Song-Takagi-Tasaki)

For a subset $\Delta \subset F$ there exists unique $H \in \mathcal{C}^{\Delta} \cap S$ such that $\mathrm{Ad}(K)H$ is a minimal submanifold in the hypersphere S.

Corollary

When $\Delta = \{\alpha_i\} \subset F$, the orbit $\mathrm{Ad}(K)H$ is isolated, so is a minimal submanifold in S.

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Construction of a retraction

$$\mathrm{Ad}(K)H \subset S \qquad (H \in \bar{\mathcal{C}},\ \|H\| = 1)$$

Then $\exists \Delta \subset F$ s.t. $H \in \mathcal{C}^{\Delta}$

Proposition

For a smooth function $f : \bar{\mathcal{C}} \to \mathbb{R}_{\geq 0}$ which satisfies

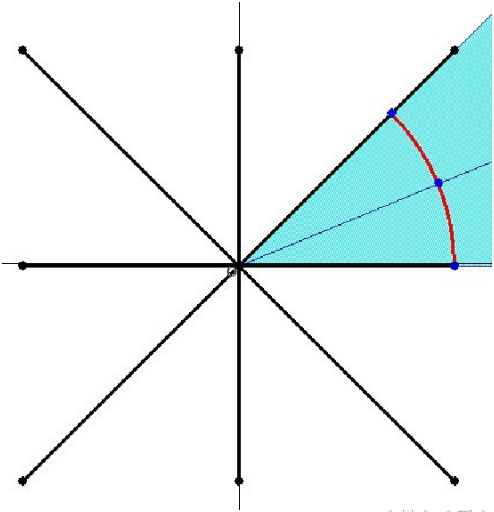
1. $f(tH) = t \quad (t \geq 0)$,
2. $f|_{\mathcal{C}^{\Delta'}} \equiv 0$ *for* $\forall \Delta' \subset F,\quad \Delta' \not\supset \Delta$,

define a mapping $\phi : \bar{\mathcal{C}} \to \{tH \mid t \geq 0\}\ ;\ x \mapsto f(x)H$.
Then ϕ extends to a retraction $\Phi : \mathfrak{m} \to C_{\mathrm{Ad}(K)H}$ as

$$\Phi(X) = \mathrm{Ad}(k)\phi(H) \qquad (\forall X \in \mathfrak{m}),$$

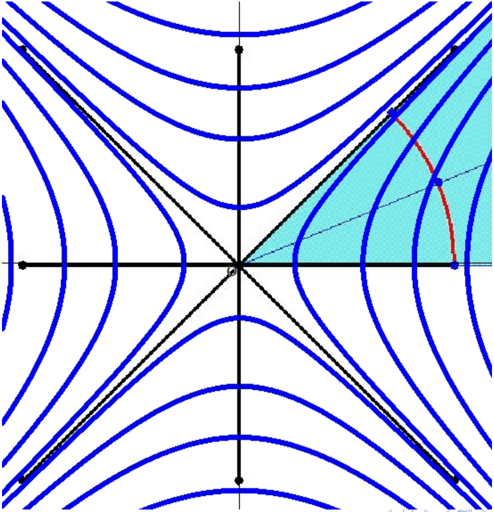
where $X = \mathrm{Ad}(k)H \quad (k \in K,\ H \in \bar{\mathcal{C}})$.
Moreover, Φ is area-nonincreasing if $J(d\Phi_x) \leq 1$ for all $x \in \mathcal{C}$.

Case of type B_2



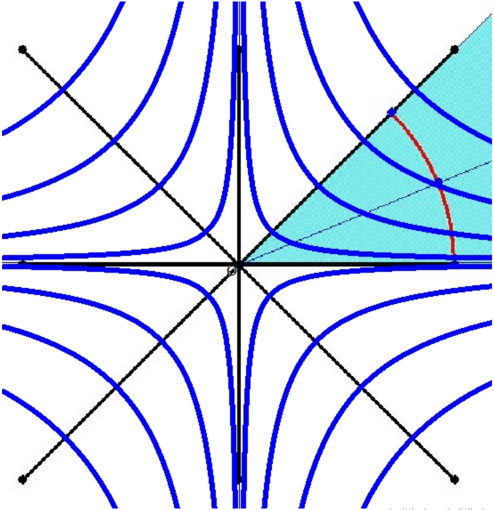
Case of type B_2

$H = e_1$ $\qquad$ $f(x) = \sqrt{\langle e_1 - e_2, x\rangle \langle e_1 - e_2, x\rangle}$



Case of type B_2

$$H = \frac{1}{\sqrt{2}}(e_1 + e_2) \qquad f(x) = \sqrt{2\langle e_1, x\rangle\langle e_2, x\rangle}$$



Calibrated geometry

$(\widetilde{M}, g)$: n-dim. Riemannian manifold

Definition

α : **calibration** *on* $\widetilde{M}$

$\overset{\text{def}}{\iff}$ α *is a closed* k*-form on* $\widetilde{M}$ *which satisfies*

for any k*-dim. subspace* $V \subset T_x\widetilde{M}$

$$\alpha|_V \leq \mathrm{vol}_V$$

$\iff$ *for any orthonormal system* $\{e_1, \ldots, e_k\}$ *of* $T_x\widetilde{M}$

$$\alpha(e_1, \ldots, e_k) \leq 1$$

Calibrated geometry

Definition

*$M \subset \widetilde{M}$: **calibrated** submanifold w.r.t. a calibration α*
$\overset{\text{def}}{\Longleftrightarrow}$ for all $x \in M$

$$\alpha|_{T_x M} = \mathrm{vol}_{T_x M}$$

Example

A complex k-dimensional complex submanifold in a Kähler manifold $(\widetilde{M}, J, \omega)$ is calibrated by $\frac{\omega^k}{k!}$.

Theorem (Harvey-Lawson)

A calibrated submanifold is volume minimizing in its homology class.

Special Lagrangian submanifolds

$(\widetilde{M}, J, \omega, \Omega)$: Calabi-Yau manifold

Definition

$L \subset \widetilde{M}$: **special Lagrangian submanifold** *of phase* θ

$\overset{\text{def}}{\Longleftrightarrow}$ L *is calibrated by* $\mathrm{Re}(e^{i\theta}\Omega)$ *for some* $\theta \in \mathbb{R}$.

$$\Longleftrightarrow \quad \begin{cases} \dim L = \frac{1}{2} \dim \widetilde{M} \\ \omega|_L \equiv 0 \\ \mathrm{Im}(e^{i\theta}\Omega|_L) \equiv 0 \end{cases}$$

Example

An n-dim. real subspace $V \subset \mathbb{C}^n$ is calibrated by $\mathrm{Re}(\Omega_0)$

$\Longleftrightarrow$ $V = gU$ for some $g \in SU(n)$

where $U = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \in \mathbb{R}\} \subset \mathbb{C}^n$

Definition of austere submanifold

Definition (Harvey-Lawson)

$M \subset \widetilde{M}$: **austere submanifold**

$\overset{\text{def}}{\Longleftrightarrow}$ *for all* $\xi \in T_x^{\perp} M$, *the set of eigenvalues with their multiplicities of the shape operator* A_ξ *of* M *is invariant under the multiplication by* -1.

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

Special Lagrangian normal bundles

$M \subset S^n$: submanifold

$$\begin{array}{rcll} \Phi : N^1M \times S^1 & \longrightarrow & S^{2n+1} \subset \mathbf{R}^{2n+2} & \text{Legendrian} \\ (x,\ \xi,\ e^{i\theta}) & \longmapsto & (\cos\theta\, x,\ \sin\theta\, \xi) & \end{array}$$

Proposition (Harvey-Lawson)

$M \subset S^n$: *austere* $\iff$ Φ : *minimal*

Borrelli-Gorodski defined a map $\widetilde{\Phi}$ modifying Φ and showed that

A_ξ does not have 0-eigenvalue $\implies$ $\widetilde{\Phi}$: Legendrian immersion

$\widetilde{\Phi}(M) \subset S^{2n+1}$ is a minimal Legendrian submanifold

$\iff C_{\widetilde{\Phi}(M)} \subset \mathbb{C}^{n+1}$ is a special Lagrangian submanifold

Definition of a weakly reflective submanifold

Definition (Ikawa-Tasaki-S.)

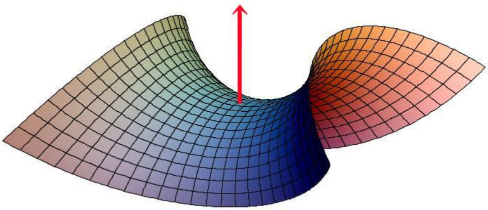
$M \subset \widetilde{M}$: **weakly reflective submanifold (WRS)**

$\overset{\text{def}}{\Longleftrightarrow}$ for each $x \in M$ and each $\xi \in T_x^\perp M$,

there exists an isometry σ_ξ of $\widetilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M.$$

We call σ_ξ a **reflection** of M with respect to ξ.



Weakly reflective submanifolds

Example

$S^n(1) \times S^n(1) \subset S^{2n+1}(\sqrt{2})$ is a weakly reflective submanifold.

Proposition

reflective $\subset$ *WRS* $\subset$ *austere* $\subset$ *minimal*

Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.

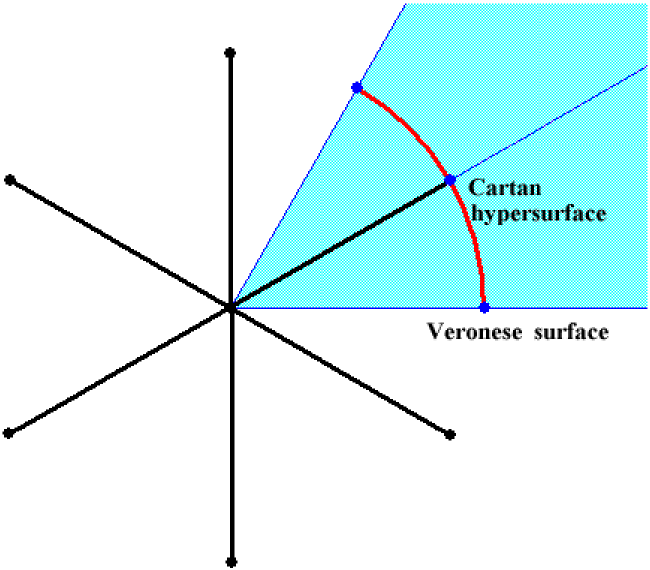
Austere orbits of s-representations (Ikawa-Tasaki-S.)

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is an austere submanifold in the hypersphere S is one of the following list:

1. An orbit through a restricted root
2. $R = A_2; \quad H = 2e_1 - e_2 - e_3, \; e_1 + e_2 - 2e_3$
3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$
4. $R = D_n; \quad H = e_1$
5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$
6. $R = B_2$ with constant multiplicities; $\quad H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$
7. $R = G_2; \quad H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$

Moreover, in the cases (1)∼(5), these austere orbits are weakly reflective submanifolds in S.

Case of type A_2



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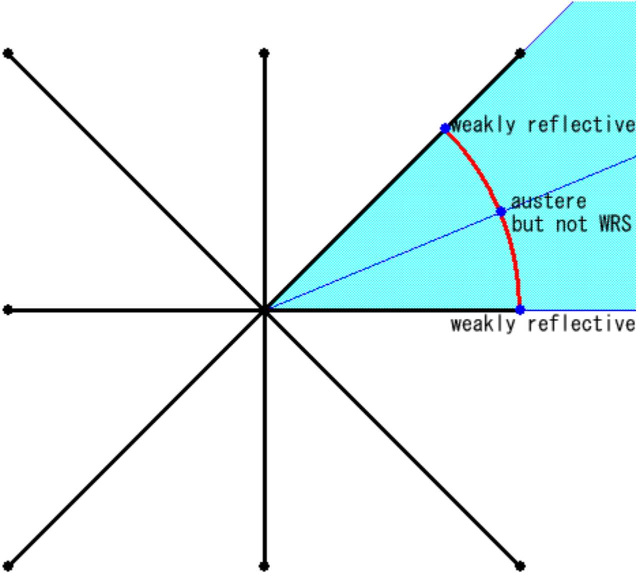

Case of type G_2