複素球面内の特殊 Lagrange 部分多様体の構成について

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Table of contents

Table of contents

Table of contents

Table of contents

Table of contents

Calabi-Yau manifolds

Definition

(M, J, ω, Ω) : **almost Calabi-Yau manifold** *of complex dim.* n

$\overset{\text{def}}{\Longleftrightarrow}$ (M, J, ω) : *Kähler manifold of complex dim.* n

Ω : *non-vanishing holomorphic* $(n, 0)$*-form on* M

(M, J, ω, Ω) : **Calabi-Yau manifold**

$\overset{\text{def}}{\Longleftrightarrow}$ *In addition*

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega}$$

Calabi-Yau manifold $\Longrightarrow$ Ricci-flat

$\overset{\pi_1=\{e\}}{\Longleftarrow}$ $\mathrm{Hol}(M, g) \subset SU(n)$

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Examples

Example (Calabi-Yau manifold)

$(\mathbb{C}^n, \sqrt{-1}, \omega_0, \Omega_0)$: complex Euclidean space

$$\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$
$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_n$$

Example (almost Calabi-Yau manifold)

P : polynomial

$$Q := \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid P(z_0) = \sum_{i=1}^{n} z_i^2 \right\}$$

$$\Omega \wedge d(-P(z_0) + z_1^2 + \cdots + z_n^2) = dz_0 \wedge \cdots \wedge dz_n$$

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Calibrated geometry

(M, g) : n-dim. Riemannian manifold

Definition

α : **calibration** *on* M

$\overset{\text{def}}{\Longleftrightarrow}$ α *is a closed* k*-form on* M *which satisfies*

for any k*-dim. subspace* $V \subset T_x M$

$$\alpha|_V \leq \mathrm{vol}_V$$

$\Longleftrightarrow$ *for any orthonormal system* $\{e_1, \ldots, e_k\}$ *of* $T_x M$

$$\alpha(e_1, \ldots, e_k) \leq 1$$

Calibrated geometry

Definition

$X \subset M$: **calibrated** *submanifold w.r.t. a calibration* α

$\overset{\text{def}}{\Longleftrightarrow}$ *for all* $x \in X$

$$\alpha|_{T_x X} = \mathrm{vol}_{T_x X}$$

Example

A complex k-dimensional complex submanifold in a Kähler manifold (M, J, ω) is calibrated by $\frac{\omega^k}{k!}$.

Theorem (Harvey-Lawson)

A calibrated submanifold is volume minimizing in its homology class.

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Special Lagrangian geometry

(M, J, ω, Ω) : Calabi-Yau manifold

Definition

$L \subset M$: **special Lagrangian submanifold** *of phase* θ

$\overset{\text{def}}{\Longleftrightarrow}$ L *is calibrated by* $\mathrm{Re}(e^{i\theta}\Omega)$ *for some* $\theta \in \mathbb{R}$.

$$\Longleftrightarrow \quad \begin{cases} \omega|_L \equiv 0 \\ \mathrm{Im}(e^{i\theta}\Omega|_L) \equiv 0 \end{cases}$$

Example

An n-dim. real subspace $V \subset \mathbb{C}^n$ is calibrated by $\mathrm{Re}(\Omega_0)$

$\Longleftrightarrow \quad V = gU$ for some $g \in SU(n)$

where $U = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \in \mathbb{R}\} \subset \mathbb{C}^n$

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Stenzel metric on T^*S^n

$$S^n = SO(n+1)/SO(n) =: G/K$$

$$T^*S^n = \left\{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi \rangle = 0\right\}$$

$$Q^n = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum_{i=0}^{n} z_i^2 = 1 \right\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$$

$$\begin{array}{rcll} \varphi : T^*S^n & \longrightarrow & Q^n \subset \mathbb{C}^{n+1} & \text{diffeomorphism} \\ (x,\xi) & \longmapsto & x\cosh(\|\xi\|) + \sqrt{-1}\dfrac{\xi}{\|\xi\|}\sinh(\|\xi\|) & \end{array}$$

- $G = SO(n+1)$ acts on T^*S^n and Q^n with cohomogeneity one.
- φ is G-equivariant.

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$$\omega_{Stz} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=0}^{n}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}u(r^2)dz_i \wedge d\bar{z}_j$$

where

$$r^2 = \|z\|^2 = \sum_{i=0}^{n} z_i \bar{z}_i$$

and u is a real function which satisfies

$$\frac{d}{d\tau}(u'(\tau))^n = cn(\sinh\tau)^{n-1}$$

where $\tau = \cosh^{-1}(r^2)$.

- When $n = 2$, Stenzel metric on T^*S^2 coincides with Eguchi-Hanson metric.

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Ω : holomorphic $(n,0)$-form on Q^n defined by

$$\Omega \wedge d(z_0^2 + z_1^2 + \cdots + z_n^2) = dz_0 \wedge dz_1 \wedge \ldots \wedge dz_n.$$

For some $\lambda \in \mathbb{C}$

$$\omega_{Stz}^n = \lambda \Omega \wedge \bar{\Omega}.$$

ω_{Stz} and Ω are invariant under $SO(n+1)$.

$(T^*S^n, J, \omega_{Stz}, \Omega)$ is a cohomogeneity one Calabi-Yau manifold.

Austere submanifold

Definition (Harvey-Lawson)

$X \subset M$: **austere submanifold**

$\overset{\text{def}}{\iff}$ *for all* $\xi \in N_x X$, *the set of eigenvalues with their multiplicities of the shape operator* A_ξ *of* X *is invariant under the multiplication by* -1.

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

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Special Lagrangian normal bundle

$X \subset S^n$: submanifold

$$\begin{aligned} \Phi : N^1X \times S^1 &\longrightarrow S^{2n+1} \subset \mathbf{R}^{2n+2} \\ (x,\ \xi,\ e^{i\theta}) &\longmapsto (\cos\theta\, x,\ \sin\theta\, \xi) \end{aligned}$$

Proposition (Harvey-Lawson)

$X \subset S^n$: *austere* $\iff$ Φ : *minimal*

Borrelli-Gorodski defined a map $\tilde{\Phi}$ modifying Φ and showed that

A_ξ does not have 0-eigenvalue $\implies$ $\tilde{\Phi}$: Legendrian immersion

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Special Lagrangian normal bundles

$X \subset S^n$: submanifold

Then $L = N^*X$ is a Lagrangian submanifold of T^*S^n with respect to the canonical symplectic structure ω_0.

Theorem (Karigiannis-Min-Oo)

$L = N^*X$ is a Lagrangian submanifold of T^*S^n with respect to Stenzel metric ω_{Stz}.
Moreover, L is a special Lagrangian submanifold of T^*S^n if and only if X is an austere submanifold in S^n.

Weakly reflective submanifold

Definition (Ikawa-Tasaki-S.)

$X \subset M$: **weakly reflective submanifold (WRS)**

$\overset{\text{def}}{\Longleftrightarrow}$ *for each* $x \in X$ *and each* $\xi \in N_x X$,

there exists an isometry σ_ξ *of* M *which satisfies*

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(X) = X.$$

We call σ_ξ *a* **reflection** *of* X *with respect to* ξ.

Example

$S^n(1) \times S^n(1) \subset S^{2n+1}(\sqrt{2})$ is a weakly reflective submanifold.

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Proposition

$$\textit{reflective} \subset \textit{WRS} \subset \textit{austere} \subset \textit{minimal}$$

Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeniety one action on a Riemannian manifold is a weakly reflective submanifold.

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Orbits of s-representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

$$K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{m} \cong T_o(G/K) : \ s\text{-representation}$$

For $H \in S \subset \mathfrak{m}$, $\quad \mathrm{Ad}(K)H \subset S$: R-space

$\mathcal{C} \subset \mathfrak{a} \subset \mathfrak{m}$: a Weyl chamber

$$\mathrm{Ad}(K)\bar{\mathcal{C}} = \mathfrak{m}$$

$\bar{\mathcal{C}} \cap S$: orbit space of $K \curvearrowright S$

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Austere orbits of s-representations (Ikawa-Tasaki-S.)

An orbit $\mathrm{Ad}(K)H$ of an irreducible s-representation which is an austere submanifold in the hypersphere S is one of the following list:

1. An orbit through a restricted root
2. $R = A_2; \quad H = 2e_1 - e_2 - e_3, \ e_1 + e_2 - 2e_3$
3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$
4. $R = D_n; \quad H = e_1$
5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$
6. $R = B_2$ with constant multiplicities; $\quad H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$
7. $R = G_2; \quad H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$

Moreover, in the cases (1)∼(5), these austere orbits are weakly reflective submanifolds in S.

Cohomogeneity one special Lagrangian submanifolds

$\gamma(s) \subset \mathbb{C}$: regular curve

$p + q = n + 1$

$$\Psi : I \times S^{p-1} \times S^{q-1} \longrightarrow Q^n \subset \mathbb{C}^n$$
$$(s, x, y) \longmapsto \left(\gamma(s)x_1, \ldots, \gamma(s)x_p, \sqrt{1-\gamma(s)^2}y_1, \ldots, \sqrt{1-\gamma(s)^2}y_q\right)$$

Then Ψ is a Lagrangian immersion with respect to ω_0 and ω_{Stz}.

If γ satisfies

$$\mathrm{Im}\left(\gamma'\gamma^{p-1}(1-\gamma^2)^{\frac{q-2}{2}}\right) = 0,$$

then $\Psi(I \times S^{p-1} \times S^{q-1})$ is a special Lagrangian submanifold of $T^*S^n \cong Q^n$ with respect to Stenzel metric ω_{Stz}, which is invariant under $SO(p) \times SO(q) \subset SO(n+1)$.

Asymptotic behaviour

$$\mathrm{Im}\left(\gamma'\gamma^{p-1}(1-\gamma^2)^{\frac{q-2}{2}}\right)=0 \quad \overset{|\gamma|\to\infty}{\rightsquigarrow} \quad \mathrm{Im}\left(\gamma'\gamma^{p-1}(\sqrt{-1}\gamma)^{q-2}\right)=0$$

q : even

$$\mathrm{Im}(\gamma'\gamma^{n-2})=0$$

$$\mathrm{Im}(\gamma^{n-1})=C$$

$$\left\{\arg(z)=\frac{k\pi}{n-1}\right\}$$

q : odd

$$\mathrm{Re}(\gamma'\gamma^{n-2})=0$$

$$\mathrm{Re}(\gamma^{n-1})=C$$

$$\left\{\arg(z)=\frac{k\pi}{n-1}+\frac{\pi}{2(n-1)}\right\}$$

$$(k=0,1,\dots,2n-1)$$

$$Q^n \quad \overset{|\xi|\to\infty}{\rightsquigarrow} \quad Q=\left\{(z_0,\dots,z_n)\in\mathbb{C}^{n+1} \;\middle|\; \sum_{i=0}^{n} z_i^2=0\right\}$$

almost Calabi-Yau

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Phase space

Case of $n = 4, \ p = 1, \ q = 4$

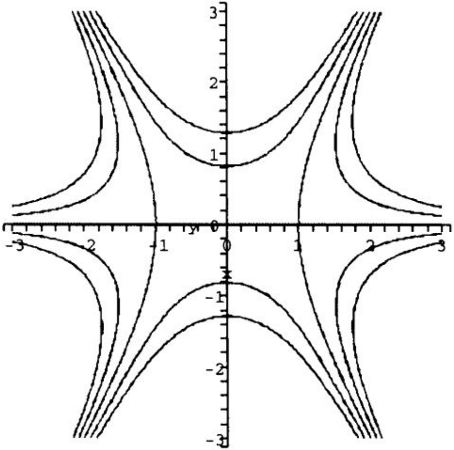

figure by H. Anciaux