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Kinematic formulas in Riemannian
homogeneous spaces

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G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

kinematic formula

$$\int_G I(M \cap gN) d\mu_G(g) = \left\{ \begin{array}{l} \text{geometric invariants} \\ \text{of } M \text{ and } N \end{array} \right\}$$

Poincaré's formula

For any curves c_1 and c_2 in \mathbb{R}^2 ,

$$\int_{I(\mathbb{R}^2)} \#(c_1 \cap gc_2) d\mu(g) = 4 \times \text{length}(c_1) \times \text{length}(c_2)$$

Generalized Poincaré formula (Howard)

G/K : Riemannian homogeneous space

$$\dim(G/K) = n$$

G : unimodular Lie group

M^p, N^q : submanifolds of G/K with $p + q \geq n$

then

$$\begin{aligned} \int_G \text{vol}(M \cap gN) d\mu_G(g) \\ = \int_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu(x, y), \end{aligned}$$

where

$$\begin{aligned} \sigma_K(T_x^\perp M, T_y^\perp N) \\ = \int_K \|u_1 \wedge \cdots \wedge u_p \wedge k_*^{-1}(v_1 \wedge \cdots \wedge v_q)\| d\mu_K(k) \end{aligned}$$

u_1, \cdots, u_p : o.n.b. of $(g_x)_*^{-1}(T_x^\perp M)$

v_1, \cdots, v_q : o.n.b. of $(g_y)_*^{-1}(T_y^\perp N)$

V_o : vector subspace of $T_o(G/K)$

$M \subset G/K$: submanifold of type V_o

$\stackrel{\text{def}}{\iff} \exists g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V_o$
for each $x \in M$

Corollary

$N \subset G/K$: submanifold of type V_o

$$\begin{aligned} \Rightarrow \int_G \text{vol}(M \cap gN) d\mu(g) \\ = \text{vol}(N) \int_M \sigma_K(T_x^\perp(M), V_o^\perp) d\mu(x) \end{aligned}$$

In addition $M \subset G/K$: submanifold of type W_o

$$\begin{aligned} \Rightarrow \int_G \text{vol}(M \cap gN) d\mu(g) \\ = \sigma_K(W_o^\perp, V_o^\perp) \text{vol}(M) \text{vol}(N) \end{aligned}$$

Theorem (Santaló)

$M^p, N^q \subset \mathbb{C}P^n$: complex submanifolds

$$\begin{aligned} \int_{U(n+1)} \text{vol}(M \cap gN) dg \\ = \frac{\text{vol}(\mathbb{C}P^{p+q-n}) \text{vol}(U(n+1))}{\text{vol}(\mathbb{C}P^p) \text{vol}(\mathbb{C}P^q)} \text{vol}(M) \text{vol}(N) \end{aligned}$$

Theorem (Kang-Takahashi-Tasaki-S.)

G/K : almost Hermitian homogeneous space

$$\dim_{\mathbb{C}}(G/K) = n$$

G : unimodular Lie group

Assume that K acts irreducibly on $\wedge^p(T_o(G/K)^{1,0})$.

Then, for any almost complex submanifolds M and N in G/K with $\dim_{\mathbb{C}}(M) = n - p$ and $\dim_{\mathbb{C}}(N) = p$,

$$\int_G \#(M \cap gN) d\mu_G(g) = \frac{\text{vol}(K)}{\binom{n}{p}} \text{vol}(M) \text{vol}(N)$$

holds.

Theorem (S.)

G/K : irreducible Hermitian symmetric space

$$\dim_{\mathbb{C}}(G/K) = n$$

Assume that K acts irreducibly on $\wedge^p(T_o(G/K)^{1,0})$

Then, for any complex submanifolds M and N in G/K with $\dim_{\mathbb{C}}(M) = n - p$, $\dim_{\mathbb{C}}(N) = n - q$ and $p + q \leq n$,

$$\begin{aligned} & \int_G \text{vol}(M \cap gN) d\mu_G(g) \\ &= \frac{(n - p)!(n - q)! \text{vol}(K)}{n!(n - p - q)!} \text{vol}(M) \text{vol}(N) \end{aligned}$$

holds.

Theorem (Chern, Federer)

$M^p, N^q \subset \mathbb{R}^n$: submanifolds

For $0 \leq 2l \leq p + q - n$,

$$\begin{aligned} \int_G \mu_{2l}(M \cap gN) d\mu_G(g) \\ = \sum_{0 \leq k \leq l} C(n, p, q, k, l) \mu_{2k}(M) \mu_{2(l-k)}(N) \end{aligned}$$

holds.

Theorem (C. S. Chen)

$M^2, N^2 \subset \mathbb{R}^3$: closed surfaces

$$\begin{aligned} \int_G \left(\int_{M \cap gN} \kappa^2 ds \right) d\mu_G(g) \\ = \pi^3 \text{vol}(N) \int_M (\|h^M\|^2 + 2(H^M)^2) d\mu_M \\ + \pi^3 \text{vol}(M) \int_N (\|h^N\|^2 + 2(H^N)^2) d\mu_N \end{aligned}$$

G/K : Riemannian homogeneous space

$V_o \subset T_o(G/K)$: vector subspace

$$\text{II}(V_o) = \left\{ h \mid h : V_o \times V_o \rightarrow V_o^\perp; \text{symmetric bilinear} \right\}$$

$$K(V_o) = \{ a \in K \mid a_* V_o = V_o \}$$

$K(V_o)$ acts on $\text{II}(V_o)$ by

$$(ah)(u, v) = a_* h(a_*^{-1}u, a_*^{-1}v) \quad (u, v \in V_o)$$

\mathcal{P} : polynomial on $\text{II}(V_o)$ invariant under $K(V_o)$

$M \subset G/K$: submanifold of type V_o

For $x \in M$, define

$$\mathcal{P}(h_x^M) := \mathcal{P}(h_o^{g_x^{-1}M}),$$

and

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M.$$

$$\text{EII}(T_o(G/K))$$

$$= \left\{ \begin{array}{l} T_o(G/K) \times T_o(G/K) \rightarrow T_o(G/K) \\ ; \text{symmetric bilinear} \end{array} \right\}$$

Theorem (Howard)

G/K : Riemannian homogeneous space

G : unimodular Lie group

$V_o, W_o \subset T_o(G/K)$: vector subspaces with

$$\dim V_o + \dim W_o \geq \dim(G/K)$$

\mathcal{P} : homogeneous polynomial on $\text{EII}(T_o(G/K))$
of degree l invariant under K s.t.

$$\int_K \sigma(V_o^\perp, k_* W_o^\perp)^{1-l} d\mu_K(k) < \infty$$

Then there exist $\{Q_\alpha, R_\alpha\}_{\alpha \in A}$ which satisfy

(1) Q_α : homogeneous polynomial on $\text{II}(V_o)$
invariant under $K(V_o)$

(2) R_α : homogeneous polynomial on $\text{II}(W_o)$
invariant under $K(W_o)$

(3) $\deg Q_\alpha + \deg R_\alpha = l$ for each $\alpha \in A$

(4) for any submanifolds M of type V_o
and N of type W_o in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_{\alpha \in A} I^{Q_\alpha}(M) I^{R_\alpha}(N)$$

holds.

Example

When G/K is a real space form and $\deg \mathcal{P} = 0$,
especially $\mathcal{P} \equiv 1$

$$\int_G \text{vol}(M \cap gN) d\mu_G(g) = C \text{vol}(M) \text{vol}(N)$$

Poincaré's formula

Transfer principle

$G/K, G'/K'$: Riemannian homogeneous spaces
of same dimensions

Assume that there exist

$\rho : K \rightarrow K'$; isomorphism

$\psi : T_o(G/K) \rightarrow T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad (\forall k \in K)$$

$\Rightarrow \{K(V_o)\text{-inv. poly. on } II(V_o)\}$

$$\cong \{K'(\psi V_o)\text{-inv. poly. on } II(\psi V_o)\}$$

\Rightarrow same type kinematic formulas hold

in G/K and G'/K'

The case of real space forms

G/K : real space form

$$K = O(T_o(G/K))$$

$$K(V_o) = O(V_o) \times O(V_o^\perp)$$

There are no homogeneous polynomials of odd degree on $\mathbb{H}(V_o)$ invariant under $K(V_o)$.

$\mathcal{W}_{2l}(h)$

$$= \sum_{\substack{1 \leq i_1, \dots, i_{2l} \leq p \\ 1 \leq j_1, \dots, j_{2l} \leq p}} \delta_{j_1, \dots, j_{2l}}^{i_1, \dots, i_{2l}} R_{i_1 i_2}^{j_1 j_2}(h) \cdots R_{i_{2l-1} i_{2l}}^{j_{2l-1} j_{2l}}(h)$$

$$= 2^l \sum \det \begin{bmatrix} h_{i_1 i_1}^{k_1} & h_{i_1 i_2}^{k_1} & \cdots & h_{i_1 i_{2l}}^{k_1} \\ h_{i_2 i_1}^{k_1} & h_{i_2 i_2}^{k_1} & \cdots & h_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_{2l-1} i_1}^{k_l} & h_{i_{2l-1} i_2}^{k_l} & \cdots & h_{i_{2l-1} i_{2l}}^{k_l} \\ h_{i_{2l} i_1}^{k_l} & h_{i_{2l} i_2}^{k_l} & \cdots & h_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}$$

$$\mu_{2l}(M) = I^{\mathcal{W}_{2l}}(M)$$

Generalized Gauss-Bonnet theorem

M : compact oriented Riemannian manifold of dimension $2l$

$$\mu_{2l}(M) = C(l)\chi(M)$$

Hotelling-Weyl tube formula

$M \subset \mathbb{R}^n$: p -dimensional submanifold

$$\text{vol}(\tau_r M) = \sum_{0 \leq 2l \leq p} C(n, p, l) \mu_{2l}(M) r^{n-p+2l}$$

The case of degree two

The space of homogeneous polynomials of degree two on $II(V_o)$ invariant under $O(V_o)$ is spanned by

$$Q_1(h) = \sum_{\substack{1 \leq i, j \leq p \\ p+1 \leq k \leq n}} (h_{ij}^k)^2 = \|h\|^2$$

$$Q_2(h) = \sum_{p+1 \leq k \leq n} \left(\sum_{1 \leq i \leq p} h_{ii}^k \right)^2 = p^2 H^2$$

$$W_2(h) = 2(Q_2(h) - Q_1(h))$$

$$U_k(h) = kQ_1(h) - Q_2(h)$$

Fact (Chern-Federer, Howard)

Assume that $2 \leq p + q - n$. Then there exist constants $a(p, q, n)$ and $b(p, q, n)$ so that for any compact submanifolds M^p and N^q of a real space from G/K the kinematic formulas

$$\begin{aligned} \int_G I^{\mathcal{W}^2}(M \cap gN) dg &= a(p, q, n) I^{\mathcal{W}^2}(M) \text{vol}(N) \\ &\quad + a(q, p, n) \text{vol}(M) I^{\mathcal{W}^2}(N) \\ \int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN) dg &= b(p, q, n) I^{\mathcal{U}_p}(M) \text{vol}(N) \\ &\quad + b(q, p, n) \text{vol}(M) I^{\mathcal{U}_q}(N) \end{aligned}$$

holds.

Theorem (Kang-Suh-S.)

In the previous fact,

$$\begin{aligned} a(p, q, n) &= \frac{p + q - n - 1}{p - 1} \\ &\quad \times \frac{\text{vol}(SO(n + 1)) \text{vol}(S^{p+q-n})}{\text{vol}(S^p) \text{vol}(S^q)}, \\ b(p, q, n) &= \frac{(p + q - n + 2)(p + q - n - 1)}{(p + 2)(p - 1)} \\ &\quad \times \frac{\text{vol}(SO(n + 1)) \text{vol}(S^{p+q-n})}{\text{vol}(S^p) \text{vol}(S^q)}. \end{aligned}$$

The case where the intersection is a curve

M^p, N^{n-p+1} : compact submanifolds of a real space form G/K of dimension n .

$$I^{\kappa^2}(M \cap gN) = \int_{M \cap gN} \kappa^2 d\sigma.$$

Proposition

There exist constants $c(p, n)$ and $d(p, n)$ s.t.

$$\begin{aligned} & \int_G I^{\kappa^2}(M \cap gN) dg \\ &= \left(c(p, n) I^{\mathcal{W}^2}(M) + d(p, n) I^{\mathcal{U}_p}(M) \right) \text{vol}(N) \\ & \quad + \text{vol}(M) \left(\begin{array}{l} c(n-p+1, n) I^{\mathcal{W}^2}(N) \\ + d(n-p+1, n) I^{\mathcal{U}_{n-p+1}}(N) \end{array} \right). \end{aligned}$$

Theorem (Kang-Suh-S.)

In the previous Proposition

$$c(p, n) = \frac{2\pi \text{vol}(SO(n+1))}{(p-1) \text{vol}(S^p) \text{vol}(S^{n-p+1})},$$

$$d(p, n) = \frac{6\pi \text{vol}(SO(n+1))}{(p+2)(p-1) \text{vol}(S^p) \text{vol}(S^{n-p+1})}.$$