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Integral geometry and Hamiltonian volume minimizing property, II

Application of integral geometry
to variational problems

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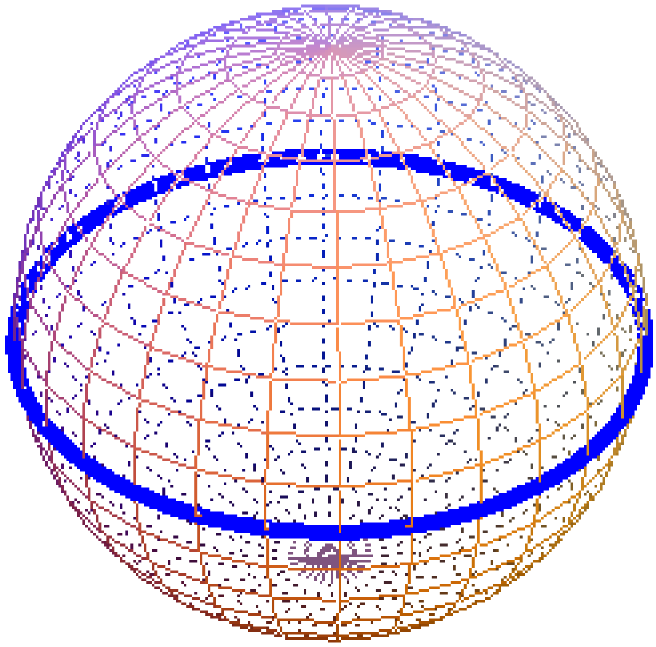
Hamiltonian stability of Lagrangian submanifolds

$S^1 \subset S^2$: equator



S^1 is an "unstable" minimal submanifold

But!

L : area-bisecting curve

$\Rightarrow \text{vol}(L) \geq \text{vol}(S^1)$

(1) $\#(S^1 \cap L) \geq 2$

(2) $\int_{SO(3)} \#(c \cap gc')d\mu(g) = 4\text{vol}(c)\text{vol}(c')$

$$\begin{aligned} 4\text{vol}(S^1)\text{vol}(L) &= \int_{SO(3)} \#(S^1 \cap gL)d\mu(g) \\ &\geq \int_{SO(3)} \#(S^1 \cap gS^1)d\mu(g) \\ &= 4\text{vol}(S^1)^2 \end{aligned}$$

(M^{2m}, ω) : symplectic manifold

$L^m \subset M$: Lagrangian submanifold

(i.e. $\omega|_{TL} \equiv 0$)

$F : M \times [0,1] \to \mathbb{R}$, C^∞

$\rightsquigarrow X_t \in \mathfrak{X}(M)$: Hamiltonian vector field

$$i(X_t)\omega = dF_t$$

$\rightsquigarrow \phi_t \in \mathrm{Diff}(M)$ s.t.

$$\frac{d\phi_t}{dt} = X_t \circ \phi_t, \quad \phi_0 = \mathrm{id}_M$$

<u>Definition</u>

$\phi \in \mathrm{Diff}(M)$: <u>Hamiltonian isotopy</u>

${}^\exists F : M \times [0,1] \to \mathbb{R}$ s.t. $\phi = \phi_1$

$$\mathrm{Ham}(M, \omega) := \{\text{Hamiltonain isotopy of } M\}$$

(M, J, ω) : Kähler manifold

$L \subset M$: Lagrangian submanifold

Definition

L : Hamiltonian minimal

$$\overset{\text{def}}{\iff} \frac{d}{dt}\bigg|_{t=0} \text{vol}(\phi_t L) = 0$$

$$^\forall\{\phi_t\}_{-\epsilon<t<\epsilon} \subset \text{Ham}(M, \omega)$$

L : Hamiltonian stable

$$\overset{\text{def}}{\iff} \frac{d^2}{dt^2}\bigg|_{t=0} \text{vol}(\phi_t L) \geq 0$$

$$^\forall\{\phi_t\}_{-\epsilon<t<\epsilon} \subset \text{Ham}(M, \omega)$$

L : Hamiltonian volume minimizing

$$\overset{\text{def}}{\iff} \text{vol}(\phi L) \geq \text{vol}(L)$$

$$^\forall\phi \in \text{Ham}(M, \omega)$$

<u>**Theorem**</u> (Oh)

(M, J, ω) : Kähler-Einstein manifold

Ricci form $\rho = c\omega$

$L \subset M$: minimal Lagrangian submanifold

Then, L is Hamiltonian stable $\Leftrightarrow \lambda_1(L) \geq c$

$\lambda_1(L)$: first eigenvalue of the Laplacian on L

Examples of Hamiltonian stable minimal Lagrangian submanifold

(1) $\mathbb{R}P^n$, Clifford torus $\subset \mathbb{C}P^n$ (Oh)

(2) G/K : cpt Hermitian symmetric space

$L \subset G/K$: real form, Einstein (Oh)

H-stabilities of non-Einstein cases were determined by Amarzaya-Ohnita

(3) $L \subset \mathbb{C}P^n$: cpt minimal Lagrangian submanifold with parallel 2nd fundamental form (Amarzaya-Ohnita)

Conjecture (Oh)

M : Kähler-Einstein manifold

τ : anti-holomorphic involutive isometry

$$L := \mathsf{Fix}(\tau)$$

Assume that L is also Einstein whose Ricci curvature is positive

$$\Rightarrow \mathrm{vol}(\phi L) \geq \mathrm{vol}(L) \quad ({}^{\forall}\phi \in \mathrm{Ham}(M))$$

Theorem (Kleiner-Oh)

$L := \mathbb{R}P^n \subset \mathbb{C}P^n$

$$\Rightarrow \mathrm{vol}(\phi L) \geq \mathrm{vol}(L) \quad ({}^{\forall}\phi \in \mathrm{Ham}(\mathbb{C}P^n))$$

Theorem (Goldstein) arXiv:math.DG0406334

(1) $\mathbb{R}P^{2m} \subset \mathbb{C}P^n$ minimizes volume among the isotropic submanifold in its $\mathbb{Z}_2$ homology class.

(2) $\mathbb{R}P^{2m-1} \subset \mathbb{C}P^n$ is Hamiltonian volume minimizing.

Theorem (Iriyeh-Ono-S.)

$L := S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$

$\Rightarrow \mathrm{vol}(\phi L) \geq \mathrm{vol}(L) \quad ({}^\forall \phi \in \mathrm{Ham}(S^2 \times S^2))$

Moreover, $\mathrm{vol}(\phi L) = \mathrm{vol}(L)$

$\Rightarrow {}^\exists\, g \in \mathrm{Isom}(S^2(1) \times S^2(1))$ s.t. $\phi L = gL$

Application of integral geometry

$N \subset G/K$: submanifold of type V_o

If $M \subset G/K$: submanifold which satisfies

(1) $\dim M + \dim N = \dim(G/K)$

(2) for almost all $x \in M$

$$\begin{aligned} & \sigma_K(T_x^\perp M, V_o^\perp) \\ = \ & \max\{\sigma_K(V, V_o^\perp) \mid V \in G_p(T_o(G/K))\} =: A \end{aligned}$$

(3) for almost all $g \in G$

$$\#(M \cap gN) = \mathrm{HIN}(M, N)$$

then, for $^\forall M' \subset G/K$ $\mathbb{Z}_2$ homologous to M,

$$\begin{aligned} \mathrm{vol}(M) &= \frac{1}{A}\int_M \sigma_K(T_x^\perp M, V_o^\perp) d\mu(x) \\ &= \frac{1}{A\mathrm{vol}(N)}\int_G \#(M \cap gN) d\mu(g) \\ &\leq \frac{1}{A\mathrm{vol}(N)}\int_G \#(M' \cap gN) d\mu(g) \\ &= \frac{1}{A}\int_{M'} \sigma_K(T_x^\perp M', V_o^\perp) d\mu(x) \\ &\leq \mathrm{vol}(M') \end{aligned}$$

Lê Hông Vân

$G_r(\mathbb{F}^{r+s}) \subset G_r(\mathbb{F}^{r+n})$ minimizes volume in its $\mathbb{Z}_2$ homology class. $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

Liu

The Pontryagin cycle $P_i \subset SO(n)$ minimizes volume in its $\mathbb{Z}_2$ homology classes.
$(i = 1, 2, \ldots, n-1)$

Tasaki

G/K : compact simply connected irreducible symmetric space

κ : maximal sectional curvature

The Helgason sphere $S^p(\kappa) \subset G/K$ minimizes volume in the non-null homotopic class

<u>**Theorem**</u> (Goldstein) arXiv:math.DG0406334

(1) $\mathbb{R}P^{2m} \subset \mathbb{C}P^n$ minimizes volume among the isotropic submanifold in its $\mathbb{Z}_2$ homology class.

(2) $\mathbb{R}P^{2m-1} \subset \mathbb{C}P^n$ is Hamiltonian volume minimizing.

Proof.

(1) even dimensional case

$$\#(\mathbb{R}P^{2m} \cap \mathbb{C}P^{n-m}) = 1$$

(a) odd dimensional case

$$\begin{array}{ccccccc}
S^{2m+1} & \supset & \psi_t(S^{2m-1}) & \longrightarrow & S_t & \subset & S^{2n+3} \\
\downarrow & & \downarrow {\scriptstyle 2:1} & & \downarrow {\scriptstyle 2:1} & & \downarrow \\
\mathbb{C}P^n & \supset & \phi_t(\mathbb{R}P^{2m-1}) & & L_t & \subset & \mathbb{C}P^{n+1}
\end{array}$$

$$S_t = \left\{ (\cos\theta x, \sin\theta) \in \mathbb{C}^{n+1} \oplus \mathbb{C} \;\middle|\; \begin{array}{l} x \in \psi_t(S^{2m-1}), \\ 0 \le \theta \le \pi \end{array} \right\}$$

Hamiltonian volume minimality

<u>Problem</u>

Does a real form of a Hermitian symmetric space of compact type which is Hamiltonian stable have Hamiltonian volume minimizing property?

Arnold-Givental inequality

G/K : cpt Hermitian symmetric space

$L \subset G/K$: real form

minimal Maslov number ≥ 2

$$\Rightarrow \#(L \cap \phi L) \geq \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2)$$

$$(\forall \phi \in \mathrm{Ham}(G/K),\ L \pitchfork \phi L)$$

If, for all Lagrangian submanifold N of G/K,

$$\int_G \#(L \cap gN)d\mu_G(g) \leq C\text{vol}(L)\text{vol}(N)$$

where $C = \dfrac{\text{vol}(G)}{\text{vol}(L)^2} \sum_i \text{rank} H_i(L, \mathbb{Z}_2)$,

then we have

$$\begin{aligned} C\text{vol}(L)\text{vol}(\phi L) &\geq \int_G \#(L \cap g \circ \phi(L))d\mu(g) \\ &\geq \int_G \sum_i \text{rank} H_i(L, \mathbb{Z}_2)d\mu(g) \\ &= \text{vol}(G) \sum_i \text{rank} H_i(L, \mathbb{Z}_2). \end{aligned}$$

Hence $\quad \text{vol}(\phi L) \geq \text{vol}(L)$.

Formulation due to Howard

$$\int_G \#(M \cap gN) d\mu(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu(x,y)$$

For $V^p, W^q \subset T_o(G/K)$

$$\sigma_K(V,W) = \int_K \sigma(V,\ k_*^{-1} W) d\mu(k)$$

σ_K is a function on $G_p(T_o(G/K)) \times G_q(T_o(G/K))$ invariant under K

$\rightsquigarrow$ σ_K is a function on the orbit space of

$K \curvearrowright G_p(T_o(G/K))$

Our program

Express σ_K concretely using the isotropy invariants, and obtain an estimation by the inequality.

Example

$G/K = \mathbb{C}P^n$

$U(n) \curvearrowright SO(2n)/SO(p) \times SO(2n-p) = \tilde{G}_p(T_o(G/K))$

Hermann action

$\rightsquigarrow$ multiple Kähler angle

The case of $S^2 \times S^2$

$$G = SO(3) \times SO(3), \quad K = SO(2) \times SO(2)$$

$$T_o(G/K) = T_{o_1}(S^2) \oplus T_{o_2}(S^2)$$

$$\begin{aligned} K \curvearrowright \tilde{G}_2(T_o(G/K)) &= SO(4)/SO(2) \times SO(2) \\ &=: G'/K' \end{aligned}$$

$$\mathrm{Ad}(K) = K'$$

Decompose $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$, and take a maximal abelian subspace $\mathfrak{a}'$ in $\mathfrak{m}'$

$$\mathfrak{a}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \theta_1, \theta_2 \in \mathbb{R} \right\}$$

then the positive roots Δ w.r.t. $\mathfrak{a}'$ is

$$\Delta = \{\theta_1 + \theta_2, \theta_1 - \theta_2\}$$

Lemma

$N, L \subset S^2 \times S^2$: Lagrange surfaces

in particular, L : product of curves in S^2

$$\Rightarrow \sigma_K(T_x^\perp N, T_y^\perp L) = 4\ \text{length}(\text{Ellip}(\sin^2\theta_x, \cos^2\theta_x))$$

where $2\theta_x - \pi/2$ is the Kähler angle of $T_x^\perp N$ w.r.t. $\omega_0 \oplus (-\omega_0)$

proof.

$$\sigma_K(T_x^\perp N, T_y^\perp L)$$
$$= \int_{SO(2)\times SO(2)} \|u_1 \wedge u_2 \wedge k^{-1}(v_1 \wedge v_2)\| d\mu(k)$$
$$= \int_{SO(2)\times SO(2)} |\langle k_1(u_1' \wedge u_2'), k_2(v_1 \wedge v_2)\rangle| dk_1 dk_2$$

$$P : \bigwedge^2 T_o(G/K) \to W \supset k_2(v_1 \wedge v_2)$$

<u>lemma</u>

$$4\pi \le \sigma_K(T_x^{\perp}N, T_y^{\perp}L) \le 16$$

In the 2nd inequality,

"=" $\iff$ N is also a product of curves in S^2

In this case

$$\frac{\mathrm{vol}(G)}{\mathrm{vol}(L)^2}\sum_i \mathrm{rank} H_i(L,\mathbb{Z}_2) = \frac{(8\pi^2)^2}{(4\pi^2)^2}\cdot 4 = 16$$

Open problems

(1) **<u>Problem</u>**

(2) Is a product of Hamiltonian volume minimizing Lagrangian submanifolds again Hamiltonian volume minimizing?

(3) Classify all Hamiltonian volume minimizing Lagrangian submanifolds in Hermitian symmetirc spaces.