The 10th International Workshop on Differential Geometry

Kyungpook National University

Friday 11 November, 2005

Transferred kinematic formulae in two point homogeneous spaces

Takashi Sakai

Department of Mathematics,

Tokyo Metropolitan University

E-mail : tsakai@comp.metro-u.ac.jp

URL : http://www.metro-u.ac.jp/˜tsakai

Contents

- Introduction and Examples

Contents

Contents

Contents

Contents

- Introduction and Examples
- Integarl invariants
- Kinematic formula and transfer principle
- Two point homogeneous spaces
- Transferred kinematic formulae in TPHS

Introduction

G/K : Riemannian homogeneous space
M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

$$\int_G I(M \cap gN) d\mu_G(g) = \left\{ \begin{array}{r} \text{geometric invariants} \\ \text{of } M \text{ and } N \end{array} \right\}$$

kinematic formula

Examples

- **Poincaré formula**
  $M^p, N^q \subset G/K$: real space form
  $$\int_G \text{vol}(M \cap gN) d\mu(g) = C(p,q,n)\text{vol}(M)\text{vol}(N)$$

Examples

- **Poincaré formula**

  $M^p, N^q \subset G/K$: real space form

$$\int_G \text{vol}(M \cap gN) d\mu(g) = C(p,q,n)\text{vol}(M)\text{vol}(N)$$

- **Chern-Federer formula**

  $M^p, N^q \subset \mathbb{R}^n$

  $0 \leq 2l \leq p + q - n$

$$\int_G \mu_{2l}(M \cap gN) d\mu_G(g) = \sum_{0 \leq k \leq l} C(n,p,q,k,l) \mu_{2k}(M) \mu_{2(l-k)}(N)$$

Integral invariants

G/K : Riemannian homogeneous space
$V \subset T_o(G/K) =: T$

Definition

$M \subset G/K$: submanifold of type V

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}g_x \in G$ s.t. $(g_x)_*^{-1}(T_xM) = V$ for each $x \in M$

Integral invariants

G/K : Riemannian homogeneous space
$V \subset T_o(G/K) =: T$

Definition

$M \subset G/K$: submanifold of <u>type V</u>
$\overset{\text{def}}{\Longleftrightarrow}$ $^{\exists} g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V$ for each $x \in M$

$$K \curvearrowright Gr_p(T)$$

Integral invariants

$$
\begin{aligned}
II(V) &= \{h : V \times V \to V^{\perp}; \text{symmetric bilinear}\} \\
K(V) &= \{k \in K \,|\, k_* V = V\}
\end{aligned}
$$

Integral invariants

$$II(V) = \{h : V \times V \to V^{\perp};\text{ symmetric bilinear}\}$$
$$K(V) = \{k \in K \mid k_* V = V\}$$

$K(V)$ acts on $II(V)$ by

$$(kh)(u, v) = k_* h(k_*^{-1} u, k_*^{-1} v) \quad (u, v \in V)$$

Integral invariants

$$II(V) = \{h : V \times V \to V^{\perp}; \text{symmetric bilinear}\}$$
$$K(V) = \{k \in K \,|\, k_* V = V\}$$

$K(V)$ acts on $II(V)$ by

$$(kh)(u,v) = k_* h(k_*^{-1}u, k_*^{-1}v) \quad (u, v \in V)$$

$\mathcal{P}$: polynomial on $II(V)$ invariant under $K(V)$
$M \subset G/K$: submanifold of type V

Integral invariants

$$II(V) = \{h : V \times V \to V^{\perp};\ \text{symmetric bilinear}\}$$
$$K(V) = \{k \in K \mid k_* V = V\}$$

$K(V)$ acts on $II(V)$ by

$$(kh)(u,v) = k_* h(k_*^{-1}u, k_*^{-1}v) \quad (u, v \in V)$$

$\mathcal{P}$: polynomial on $II(V)$ invariant under $K(V)$
$M \subset G/K$: submanifold of type V

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M$$

Integral invariants

$$EII(T) = \{H : T \times T \to T; \text{symmetric bilinear}\}$$

K acts on $EII(T)$

Integral invariants

$$EII(T) = \{H : T \times T \to T; \text{symmetric bilinear}\}$$

K acts on $EII(T)$

$\mathcal{P}$: polynomial on $EII(T)$ invariant under K
$M \subset G/K$: submanifold

$$H_x^M(u, v) := h_x^M(Pu, Pv) \qquad (u, v \in T)$$

$$P : T \to V$$

Integral invariants

$$EII(T) = \{H : T \times T \to T; \text{symmetric bilinear}\}$$

K acts on $EII(T)$

$\mathcal{P}$: polynomial on $EII(T)$ invariant under K
$M \subset G/K$: submanifold

$$H_x^M(u,v) := h_x^M(Pu, Pv) \qquad (u, v \in T)$$

$$P : T \to V$$

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(H_x^M) d\mu_M$$

Kinematic formula (Howard)

G/K : Riem. homog. space, G : unimodular
$V, W \subset T$, $\dim V + \dim W \geq \dim T$
$\mathcal{P}$: K-inv. homog. poly. on $EII(T)$
Then there exist finite pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ s.t.

(1) $\mathcal{Q}_\alpha$: $K(V)$-inv. homog. poly. on $II(V)$

(2) $\mathcal{R}_\alpha$: $K(W)$-inv. homog. poly. on $II(W)$

(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = \deg \mathcal{P}$ for each α

(4) for any submanifolds M of type V and N of type W in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$\rho \ : \ K \to K'$; isomorphism

$\psi \ : \ T_o(G/K) \to T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$\rho \ : \ K \to K'$; isomorphism

$\psi \ : \ T_o(G/K) \to T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

$$\Rightarrow \left\{ \begin{array}{r} K(V)\text{-inv. poly.} \\ \text{on } II(V) \end{array} \right\} \overset{\psi}{\cong} \left\{ \begin{array}{r} K'(\psi V)\text{-inv. poly.} \\ \text{on } II(\psi V) \end{array} \right\}$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

ρ : $K \to K'$; isomorphism

ψ : $T_o(G/K) \to T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

$$\Rightarrow \left\{ \begin{array}{r} K(V)\text{-inv. poly.} \\ \text{on } II(V) \end{array} \right\} \overset{\psi}{\cong} \left\{ \begin{array}{r} K'(\psi V)\text{-inv. poly.} \\ \text{on } II(\psi V) \end{array} \right\}$$

$$\Rightarrow \int_{G'} I^{\mathcal{P}'}(M' \cap gN')d\mu_{G'}(g) = \sum_\alpha I^{\mathcal{Q}'_\alpha}(M')I^{\mathcal{R}'_\alpha}(N')$$

holds for M' of type ψV and N' of type ψW in G'/K'.

Two point homogeneous spaces

M : two point homogeneous space (TPHS)

$\overset{\text{def}}{\Longleftrightarrow}$ $x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
$^\exists g \in \text{Isom}(M)$ s.t. $gx_1 = x_2,\ gy_1 = y_2$

Two point homogeneous spaces

M : <u>two point homogeneous space</u> (TPHS)

$\overset{\text{def}}{\Longleftrightarrow}$ $x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
$^{\exists}g \in \text{Isom}(M)$ s.t. $gx_1 = x_2,\ gy_1 = y_2$

M : <u>isotropic</u>

$\overset{\text{def}}{\Longleftrightarrow}$ $\text{Isom}(M)_x = \{g \in \text{Isom}(M) \mid gx = x\}$
acts transitively on the unit sphere in T_xM,
for every $x \in M$

Two point homogeneous spaces

M : two point homogeneous space (TPHS)

$\overset{\text{def}}{\Longleftrightarrow}$ $x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
$^\exists g \in \text{Isom}(M)$ s.t. $gx_1 = x_2,\ gy_1 = y_2$

M : isotropic

$\overset{\text{def}}{\Longleftrightarrow}$ $\text{Isom}(M)_x = \{g \in \text{Isom}(M) \mid gx = x\}$
acts transitively on the unit sphere in T_xM,
for every $x \in M$

M : TPHS $\Longleftrightarrow$ M : isotropic

$\Longleftrightarrow$ M : Euclidean space or rank 1 symmetric space

Poincaré formula in TPHS

G/K : TPHS, $\dim(G/K) = n$
$M^p, N^{n-1} \subset G/K$

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}\mathrm{vol}(M)\mathrm{vol}(N)$$

Poincaré formula in TPHS

G/K : TPHS, $\dim(G/K) = n$
$M^p, N^{n-1} \subset G/K$

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})} \mathrm{vol}(M)\mathrm{vol}(N)$$

Lemma

If G/K is a TPHS, then there is no homogeneous polynomial on $II(V)$ (resp. $EII(T)$) of odd degree invariant under $K(V)$ (resp. K).

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1\le i_1,\dots,i_{2l}\le p \\ p+1\le k_1,\dots,k_l\le n}} \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \dots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \dots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \dots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \dots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}$$

$O(V) \times O(V^\perp)$-inv. homog. poly. on $II(V)$ of degree $2l$

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \le i_1, \ldots, i_{2l} \le p \\ p+1 \le k_1, \ldots, k_l \le n}} \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \ldots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \ldots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \ldots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \ldots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}$$

$O(V) \times O(V^\perp)$-inv. homog. poly. on $II(V)$ of degree $2l$

$$\mu_{2l}(M) := I^{\mathcal{W}_{2l}}(M)$$

- **Hotelling-Weyl tube formula**
  $M \subset \mathbb{R}^n$: p-dimensional submanifold

$$\text{vol}(\tau_r M) = \sum_{0 \le 2l \le p} C(n, p, l) \mu_{2l}(M) r^{n-p+2l}$$

- **Hotelling-Weyl tube formula**
  $M \subset \mathbb{R}^n$: p-dimensional submanifold

$$\text{vol}(\tau_r M) = \sum_{0 \le 2l \le p} C(n,p,l)\mu_{2l}(M) r^{n-p+2l}$$

- **Generalized Gauss-Bonnet theorem**
  M : compact oriented Riemannian manifold of dimension $2l$

$$\mu_{2l}(M) = C(l)\chi(M)$$

The case of degree 2

The space of homogeneous polynomials of degree 2 on $II(V)$ invariant under $O(V) \times O(V^\perp)$ is spaned by

$$\mathcal{Q}_1(h) = \sum_{\substack{1 \le i,j \le p \\ p+1 \le k \le n}} \left(h_{ij}^k\right)^2 = \|h\|^2$$

$$\mathcal{Q}_2(h) = \sum_{p+1 \le k \le n} \left(\sum_{1 \le i \le p} h_{ii}^k \right)^2 = p^2 H^2$$

The case of degree 2

The space of homogeneous polynomials of degree 2 on $II(V)$ invariant under $O(V) \times O(V^{\perp})$ is spaned by

$$\mathcal{Q}_1(h) = \sum_{\substack{1\le i,j\le p\\ p+1\le k\le n}} \left(h_{ij}^k\right)^2 = \|h\|^2$$

$$\mathcal{Q}_2(h) = \sum_{p+1\le k\le n} \left(\sum_{1\le i\le p} h_{ii}^k \right)^2 = p^2 H^2$$

$$\mathcal{W}_2(h) = \mathcal{Q}_2(h) - \mathcal{Q}_1(h)$$

$$\mathcal{U}_p(h) = p\mathcal{Q}_1(h) - \mathcal{Q}_2(h)$$

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1 \leq k \leq n} \sum_{i<j} \big(\kappa_i(\xi_k) - \kappa_j(\xi_k)\big)^2$$

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1 \le k \le n} \sum_{i<j} \left(\kappa_i(\xi_k) - \kappa_j(\xi_k)\right)^2$$

$$\langle\ ,\ \rangle^* = \rho^2 \langle\ ,\ \rangle \quad \Longrightarrow \quad \kappa_i^*(\xi_k^*) = \rho^{-1}(\kappa_i(\xi_k) - \lambda_k)$$

$\mathcal{U}_p(h)\langle\ ,\ \rangle$: **conformal invariant**

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1\le k\le n} \sum_{i<j} \left(\kappa_i(\xi_k) - \kappa_j(\xi_k)\right)^2$$

$$\langle\ ,\ \rangle^* = \rho^2 \langle\ ,\ \rangle \quad \Longrightarrow \quad \kappa_i^*(\xi_k^*) = \rho^{-1}(\kappa_i(\xi_k) - \lambda_k)$$

$\mathcal{U}_p(h)\langle\ ,\ \rangle$: **conformal invariant**

Willmore-Chen functional

$$I^{\mathcal{U}_p^{p/2}}(M) := \int_M \left(\mathcal{U}_p(h_x^M)\right)^{p/2} d\mu_M(x)$$

Proposition (Chern-Federer, Howard, Kang-Suh-S.)

G/K : real space form

$M^p, N^q \subset G/K, \quad 2 \leq p+q-n$

$$\int_G I^{\mathcal{W}_2}(M \cap gN)dg = a(p,q,n)I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + a(q,p,n)\mathrm{vol}(M)I^{\mathcal{W}_2}(N)$$

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN)dg = b(p,q,n)I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + b(q,p,n)\mathrm{vol}(M)I^{\mathcal{U}_q}(N)$$

Transferred kinematic formulae

Let M and N be real hypersurfaces in a two point homogeneous space G/K. Then the following kinematic formulae hold:

$$\int_G I^{\mathcal{W}_2}(M \cap gN)\,d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n)\big(I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{W}_2}(N)\big)$$

$$\int_G I^{\mathcal{U}_{n-2}}(M \cap gN)\,d\mu_G(g) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n)\big(I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{U}_{n-1}}(N)\big)$$

Generalized kinematic formula (Howard)

G/K : Riem. homog. space, G : unimodular
$V, W \subset T$, $\dim V + \dim W \geq \dim T$
$\mathcal{P}$: K-inv. homog. poly. on $EII(T)$

Then

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu(g) = \int_{M \times N} I^{\mathcal{P}}_K(V, h^M_x, W, h^N_y) d\mu(x, y)$$

holds for any submanifolds M of type V and N of type W in G/K.

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

$^{\exists}\mathcal{Q}$: $K(V)$-inv. homog. poly. on $II(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \mathcal{Q}(h_1) + \mathcal{Q}(h(r))$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

${}^{\exists}\mathcal{Q}$: $K(V)$-inv. homog. poly. on $II(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \mathcal{Q}(h_1) + \mathcal{Q}(h(r))$$

$$I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$
$$= \; a(n-1, n-1, n) \left(\mathcal{W}_2(h_1) + \mathcal{W}_2(h(r))\right)$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

${}^{\exists}\mathcal{Q}$: $K(V)$-inv. homog. poly. on $II(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \mathcal{Q}(h_1) + \mathcal{Q}(h(r))$$

$$\begin{aligned} & I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r)) \\ & \quad = a(n-1, n-1, n)\left(\mathcal{W}_2(h_1) + \mathcal{W}_2(h(r))\right) \end{aligned}$$

$$\therefore \mathcal{Q} = \frac{\text{vol}(K)}{\text{vol}(SO(n))} a(n-1, n-1, n)\mathcal{W}_2$$

Problems

Problem 1

Can all kinematic formulae for hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$-invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

Problems

Problem 1

Can all kinematic formulae for hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$-invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

Problem 2

Does Theorem hold for a real hypersurface N and any dimensional submanifold M?