

Absolutely area-minimizing cones over some minimal submanifolds in the sphere

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- ③ Weakly reflective submanifolds and austere submanifolds
- ④ Special Lagrangian cones

Methods to show area-minimizing property

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- 1 Motivation and history of area-minimizing cones
- 2 Area-minimizing cones over some minimal orbits of s -representations
- 3 Weakly reflective submanifolds and austere submanifolds
- 4 Special Lagrangian cones

Introduction and Motivation

Plateau problem

For given a Jordan curve Γ in \mathbb{R}^n , does there exist an area-minimizing surface with boundary Γ ?

Answer (Douglas and Rado)

YES! by geometric measure theory

But such a surface may have singularities

“integral current”

Theorem (Osseman)

An area-minimizing surface M^2 in \mathbb{R}^3 with smooth boundary has no singularities.

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Theorem (Almgren)

The dimension of the singular set of an area-minimizing submanifold M^k is equal or less than $k - 2$.

An area-minimizing surface M^2 in \mathbb{R}^n ($n \geq 4$) may have 0-dimensional singular set, i.e. isolated singularities.

Questions

- *What kind of singularities can appear on area-minimizing submanifolds?*
- *Study area-minimizing cones, which are tangent cones at isolated singularities.*

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Area-minimizing cones

$$M^{k-1} \subset S^{n-1}(1) \subset \mathbb{R}^n$$

$$C_M = \{tx \mid x \in M, t \geq 0\} \subset \mathbb{R}^n$$

$$C_M^1 = \{tx \mid x \in M, 0 \leq t \leq 1\}$$

Definition

C_M is said to be **area-minimizing** if C_M^1 has minimum volume among all submanifolds with boundary M .

Proposition

$$M \subset S^{n-1} : \text{minimal} \iff C_M \subset \mathbb{R}^n : \text{minimal}$$

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History and Examples of area-minimizing cones

1969 (Bombieri-DeGiorgi-Giusti) $M = S^n \times S^n \subset S^{2n+1}$ ($n \geq 3$),
then $C_M \subset \mathbb{R}^{2n+2}$ is area-minimizing.

1972 (Lawson) $M = S^k \times S^l \subset S^{k+l+1}$ ($k+l \geq 7$),
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$M = S^2 \times S^4 \subset S^7$, then $C_M \subset \mathbb{R}^8$ is area-minimizing.

$M = S^1 \times S^5 \subset S^7$, then $C_M \subset \mathbb{R}^8$ is **NOT** area-minimizing.

1991 (Lawlor) provided a criterion for a cone to be area-minimizing,
and classified all area-minimizing cones over product of spheres.

1994 (Kerckhove), 2000 (Hirohashi-Kanno-Tasaki), 2002 (Kanno)
Cones over some symmetric R -spaces are area-minimizing.

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Area-nonincreasing retraction

$\Phi : \mathbb{R}^n \longrightarrow C_M$ differentiable retraction

is **area-nonincreasing** if

$$J(d\Phi)_x = \sup\{\|(d\Phi)_x(u_1) \wedge \cdots \wedge (d\Phi)_x(u_k)\|\} \leq 1$$

for all $x \in \mathbb{R}^n$, where u_1, \dots, u_k are orthonormal vectors of $T_x(\mathbb{R}^n)$

Proposition

If there exists an area-nonincreasing retraction $\Phi : \mathbb{R}^n \longrightarrow C_M$, then C_M is area-minimizing.

Proof. $\text{Vol}(C_M^1) \leq \text{Vol}(\Phi(N)) \leq \text{Vol}(N),$

for any submanifold N with boundary M , since $C_M^1 \subset \Phi(N)$

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Orbits of s -representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

$$K \overset{\text{Ad}}{\curvearrowright} \mathfrak{m} \cong T_o(G/K) : s\text{-representation}$$

For $H \in S \subset \mathfrak{m}$, $\text{Ad}(K)H \subset S$: R -space

$\mathfrak{a} \subset \mathfrak{m}$: maximal abelian subspace

R : restricted root system of $(\mathfrak{g}, \mathfrak{k})$

F : fundamental system of R

$$\mathcal{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\forall \alpha \in F)\} : \text{Weyl chamber}$$

$$\text{Ad}(K)\bar{\mathcal{C}} = \mathfrak{m} \rightsquigarrow \boxed{\bar{\mathcal{C}} \cap S : \text{orbit space of } K \curvearrowright S}$$

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Orbits of s -representations

For $\Delta \subset F$

$$\mathcal{C}^\Delta = \left\{ H \in \bar{\mathcal{C}} \mid \begin{array}{l} \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \\ \langle \beta, H \rangle = 0 \ (\beta \in F - \Delta) \end{array} \right\}$$

$$\bar{\mathcal{C}} \cap S = \bigcup_{\Delta \subset F} (\mathcal{C}^\Delta \cap S) \quad \text{disjoint union}$$

Theorem (Hirohashi-Song-Takagi-Tasaki)

For a subset $\Delta \subset F$ there exists unique $H \in \mathcal{C}^\Delta \cap S$ such that $\text{Ad}(K)H$ is a minimal submanifold in the hypersphere S .

Corollary

When $\Delta = \{\alpha_i\} \subset F$, the orbit $\text{Ad}(K)H$ is isolated, so is a minimal submanifold in S .

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Construction of a retraction

$$\text{Ad}(K)H \subset S \quad (H \in \bar{\mathcal{C}}, \|H\| = 1)$$

Then $\exists \Delta \subset F$ s.t. $H \in \mathcal{C}^\Delta$

Proposition

For a smooth function $f : \bar{\mathcal{C}} \rightarrow \mathbb{R}_{\geq 0}$ which satisfies

- 1 $f(tH) = t \quad (t \geq 0)$,
- 2 $f|_{\mathcal{C}^{\Delta'}} \equiv 0 \quad \text{for} \quad \forall \Delta' \subset F, \quad \Delta' \not\supset \Delta,$

define a mapping $\phi : \bar{\mathcal{C}} \rightarrow \{tH \mid t \geq 0\}$; $x \mapsto f(x)H$.

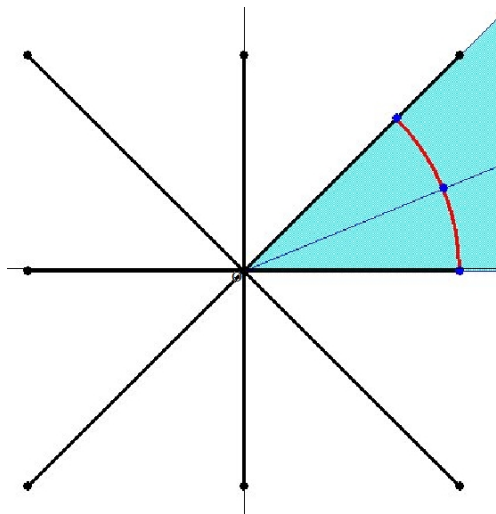
Then ϕ extends to a retraction $\Phi : \mathfrak{m} \rightarrow C_{\text{Ad}(K)H}$ as

$$\Phi(X) = \text{Ad}(k)\phi(Y) \quad (\forall X \in \mathfrak{m}),$$

where $X = \text{Ad}(k)Y \quad (k \in K, Y \in \bar{\mathcal{C}})$.

Moreover, Φ is area-nonincreasing if $J(d\Phi_x) \leq 1$ for all $x \in \mathcal{C}$.

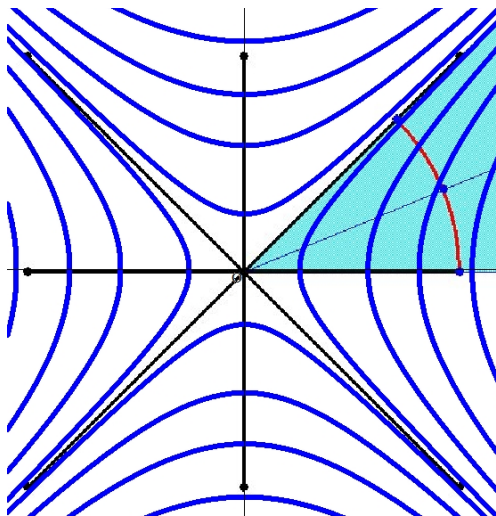
Case of type B_2



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$$H = e_1$$

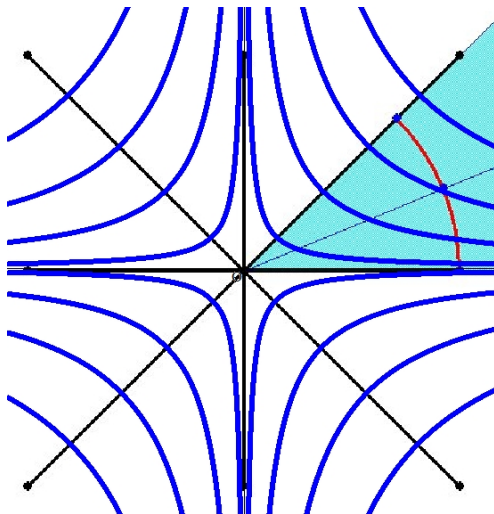
$$f(x) = \sqrt{\langle e_1 - e_2, x \rangle \langle e_1 + e_2, x \rangle}$$



Case of type B_2

$$H = \frac{1}{\sqrt{2}}(e_1 + e_2)$$

$$f(x) = \sqrt{2\langle e_1, x \rangle \langle e_2, x \rangle}$$



(\widetilde{M}, g) : n -dim. Riemannian manifold

Definition

α : **calibration** on \widetilde{M}

$\stackrel{\text{def}}{\iff} \alpha$ is a closed k -form on \widetilde{M} which satisfies
for any k -dim. subspace $V \subset T_x \widetilde{M}$

$$\alpha|_V \leq \text{vol}_V$$

\iff for any orthonormal system $\{e_1, \dots, e_k\}$ of $T_x \widetilde{M}$

$$\alpha(e_1, \dots, e_k) \leq 1$$

Definition

$M \subset \widetilde{M}$: **calibrated** submanifold w.r.t. a calibration α
 $\stackrel{\text{def}}{\iff}$ for all $x \in M$

$$\alpha|_{T_x M} = \text{vol}_{T_x M}$$

Example

A complex k -dimensional complex submanifold in a Kähler manifold $(\widetilde{M}, J, \omega)$ is calibrated by $\frac{\omega^k}{k!}$.

Theorem (Harvey-Lawson)

A calibrated submanifold is volume minimizing in its homology class.

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Special Lagrangian submanifolds

$(\widetilde{M}, J, \omega, \Omega)$: Calabi-Yau manifold

Definition

$L \subset \widetilde{M}$: **special Lagrangian submanifold** of phase θ

$\stackrel{\text{def}}{\iff} L$ is calibrated by $\text{Re}(e^{i\theta}\Omega)$ for some $\theta \in \mathbb{R}$.

$$\iff \begin{cases} \dim L = \frac{1}{2} \dim \widetilde{M} \\ \omega|_L \equiv 0 \\ \text{Im}(e^{i\theta}\Omega|_L) \equiv 0 \end{cases}$$

Example

An n -dim. real subspace $V \subset \mathbb{C}^n$ is calibrated by $\text{Re}(\Omega_0)$

$$\iff V = gU \quad \text{for some } g \in SU(n)$$

where $U = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \in \mathbb{R}\} \subset \mathbb{C}^n$

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Definition of austere submanifold

Definition (Harvey-Lawson)

$M \subset \widetilde{M}$: **austere submanifold**

$\stackrel{\text{def}}{\iff}$ for all $\xi \in T_x^\perp M$, the set of eigenvalues with their multiplicities of the shape operator A_ξ of M is invariant under the multiplication by -1 .

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

Definition of a weakly reflective submanifold

Definition (Ikawa-Tasaki-S.)

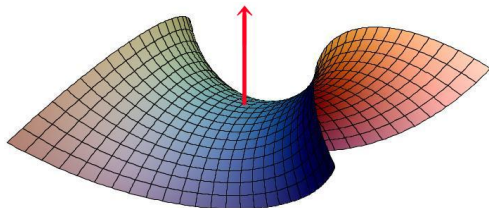
$M \subset \widetilde{M}$: **weakly reflective submanifold (WRS)**

$\stackrel{\text{def}}{\iff}$ for **each** $x \in M$ and **each** $\xi \in T_x^\perp M$,

there exists an isometry σ_ξ of \widetilde{M} which satisfies

$$\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(M) = M.$$

We call σ_ξ a **reflection** of M with respect to ξ .



Weakly reflective submanifolds

Example

$S^n(1) \times S^n(1) \subset S^{2n+1}(\sqrt{2})$ is a weakly reflective submanifold.

Proposition

reflective \subset *WRS* \subset *austere* \subset *minimal*

Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.

Special Lagrangian normal bundles

$M \subset S^n$: submanifold

$$\begin{aligned}\Phi : N^1 M \times S^1 &\longrightarrow S^{2n+1} \subset \mathbf{R}^{2n+2} && \text{Legendrian} \\ (x, \xi, e^{i\theta}) &\longmapsto (\cos \theta x, \sin \theta \xi)\end{aligned}$$

Proposition (Harvey-Lawson)

$$M \subset S^n : \textit{austere} \iff \Phi : \textit{minimal}$$

Borrelli-Gorodski defined a map $\tilde{\Phi}$ modifying Φ and showed that

$$A_\xi \text{ does not have } 0\text{-eigenvalue} \implies \tilde{\Phi} : \text{Legendrian immersion}$$

$\tilde{\Phi}(M) \subset S^{2n+1}$ is a minimal Legendrian submanifold

$$\iff C_{\tilde{\Phi}(M)} \subset \mathbb{C}^{n+1} \text{ is a special Lagrangian submanifold}$$



Stenzel metric on T^*S^n

$$S^n = SO(n+1)/SO(n) =: G/K$$

$$T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi \rangle = 0\}$$

$$Q^n = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n z_i^2 = 1 \right\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$$

$$\begin{aligned} \varphi : T^*S^n &\longrightarrow Q^n \subset \mathbb{C}^{n+1} && \text{diffeomorphism} \\ (x, \xi) &\longmapsto x \cosh(\|\xi\|) + \sqrt{-1} \frac{\xi}{\|\xi\|} \sinh(\|\xi\|) \end{aligned}$$

- $G = SO(n+1)$ acts on T^*S^n and Q^n with cohomogeneity one.
- φ is G -equivariant.

$$\omega_{Stz} = \sqrt{-1} \partial \bar{\partial} u(r^2) = \sqrt{-1} \sum_{i,j=0}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(r^2) dz_i \wedge d\bar{z}_j$$

where

$$r^2 = \|z\|^2 = \sum_{i=0}^n z_i \bar{z}_i$$

and u is a real function which satisfies

$$\frac{d}{d\tau} (u'(\tau))^n = cn(\sinh \tau)^{n-1}$$

where $\tau = \cosh^{-1}(r^2)$.

- When $n = 2$, Stenzel metric on T^*S^2 coincides with Eguchi-Hanson metric.

Ω : holomorphic $(n, 0)$ -form on Q^n defined by

$$\Omega \wedge d(z_0^2 + z_1^2 + \cdots + z_n^2) = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n.$$

For some $\lambda \in \mathbb{C}$

$$\omega_{Stz}^n = \lambda \Omega \wedge \bar{\Omega}.$$

ω_{Stz} and Ω are invariant under $SO(n+1)$.

$(T^*S^n, J, \omega_{Stz}, \Omega)$ is a cohomogeneity one Calabi-Yau manifold.

Special Lagrangian normal bundles

$X \subset S^n$: submanifold

Then $L = N^*X$ is a Lagrangian submanifold of T^*S^n with respect to the symplectic structure ω_0 induced from $T^*S^n \cong Q^n \subset \mathbb{C}^{n+1}$.

Theorem (Karigiannis-Min-Oo)

*$L = N^*X$ is a Lagrangian submanifold of T^*S^n with respect to Stenzel metric ω_{Stz} .*

*Moreover, L is a special Lagrangian submanifold of T^*S^n if and only if X is an austere submanifold in S^n .*

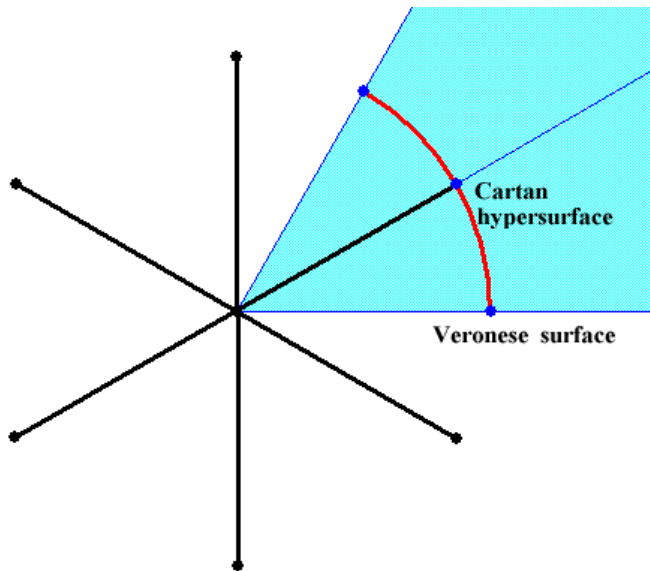
Austere orbits of s -representations (Ikawa-Tasaki-S.)

An orbit $\text{Ad}(K)H$ of an irreducible s -representation which is an austere submanifold in the hypersphere S is one of the following list:

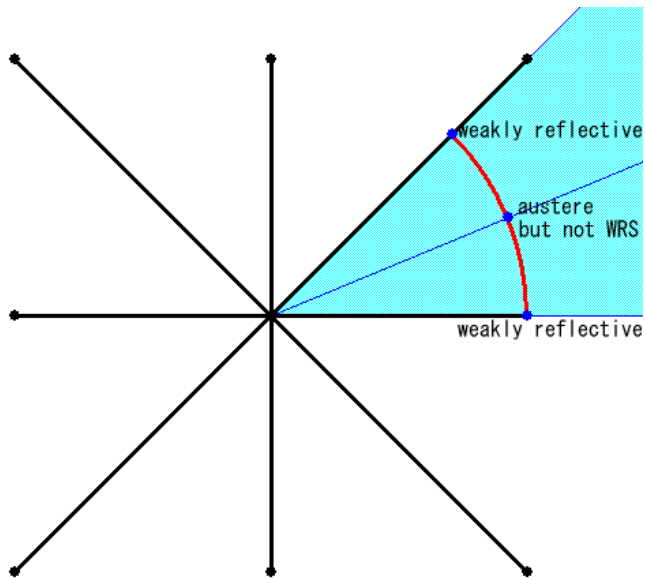
- 1 An orbit through a restricted root
- 2 $R = A_2$; $H = 2e_1 - e_2 - e_3, e_1 + e_2 - 2e_3$
- 3 $R = A_3$; $H = e_1 + e_2 - e_3 - e_4$
- 4 $R = D_n$; $H = e_1$
- 5 $R = D_4$; $H = e_1 + e_2 + e_3 \pm e_4$
- 6 $R = B_2$ with constant multiplicities; $H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}$
- 7 $R = G_2$; $H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$

Moreover, in the cases (1) ~ (5), these austere orbits are weakly reflective submanifolds in S .

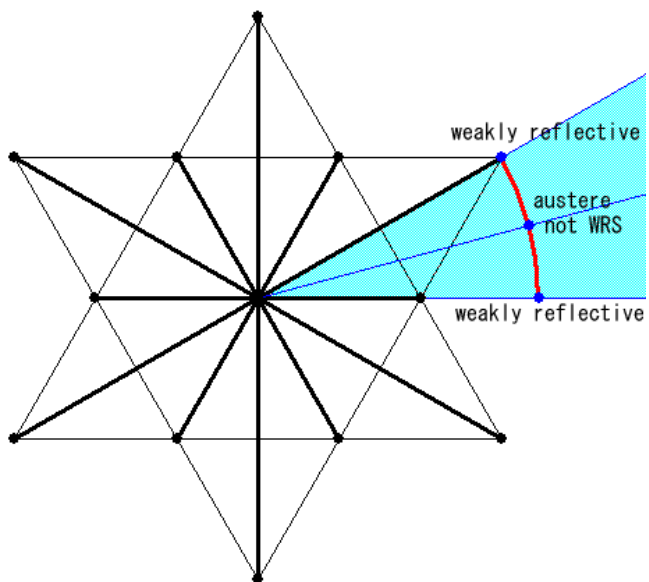
Case of type A_2



Case of type B_2



Case of type G_2



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- ② 極小軌道上の錐の面積最小性を決定せよ . 特に , austere 軌道・弱鏡映軌道の場合はどうなるか ?
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- ④ 階数 1 のコンパクト対称空間 G/K の austere 部分多様体から $T^*(G/K)$ の Stenzel 計量に関する特殊 Lagrange 部分多様体を構成せよ .

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